1. Roy is starting a baking company and decides that he will sell cupcakes. He sells \( n \) cupcakes for \((n + 20)(n + 15)\) cents. A man walks in and buys $10.50 worth of cupcakes. Roy bakes cupcakes at a rate of 10 cupcakes an hour. How many minutes will it take Roy to complete the order?

**Solution:**

Solving for \( x \), we have 
\[
(n + 20)(n + 15) = 1050 \implies x^2 + 35x - 750 = (x + 50)(x - 15) = 0
\]
and since \( x \geq 0 \), \( x = 15 \) cupcakes. So it will take Roy 1.5 hours or 90 minutes.

*Author: Heesu Hwang*

2. Let \( f \) be a function which takes in 0, 1, 2 and returns 0, 1, or 2. The values need not be distinct: for instance we could have \( f(0) = 1, f(1) = 1, f(2) = 2 \). How many such functions are there which satisfy:

\[
f(2) + f(f(0)) + f(f(f(1))) = 5
\]

**Solution:**

Since \( f \) must return 0, 1, or 2, we know that of the three values \( f(2), f(f(0)), f(f(f(1))) \), 2 of them must be 2 and one must be 1 because that is the only way to obtain 5.

**Case 1:** \( f(2) = 1 \). Then \( f(f(0)) = 2 \) and we can verify that this means that \( f(0) = 1 \) and \( f(1) = 2 \). And then \( f(f(f(1))) = 2 \) as required. So this is 1 function.

**Case 2:** \( f(f(0)) = 1 \). Then \( f(2) = 2 \). If \( f(0) = 0 \), then \( f(f(0)) = 0 \) so we must have \( f(0) = 1 \) and \( f(1) = 1 \). But then \( f(f(f(1))) = 1 \) and \( f(2) + f(f(0)) + f(f(f(1))) = 4 \) so there are no functions with \( f(f(0)) = 1 \).

**Case 3:** \( f(f(f(1))) = 1 \). Then \( f(2) = 2 \) and \( f(f(0)) = 2 \). So \( f(0) \) is 1 or 2. If \( f(0) = 1 \), then \( f(1) = f(f(0)) = 2 \) and \( f(f(f(1))) = 2 \) which isn’t the case. So \( f(0) = 2 \) as well. Then \( f(f(f(1))) = 1 \) and this can only happen if \( f(1) = 1 \). So this is another such function.

In total there are only 2 such functions, namely \((f(0), f(1), f(2)) = (1, 2, 1), (2, 1, 2)\).

*Author: Roy Zhao*

3. Andrew and Blair are bored in class and decide to play a game. They pick a pair \((a, b)\) with \(1 \leq a, b \leq 100\). Andrew says the next number in the geometric series that begins with \(a, b\) and Blair says the next number in the arithmetic series that begins with \(a, b\). For how many pairs \((a, b)\) is Andrew’s number minus Blair’s number a positive perfect square?

**Solution:**

Andrew will say \( \frac{b^2}{a} \) and Blair will say \( 2b - a \) and hence the difference will be \( \frac{b^2 - 2ab + a^2}{a} = \frac{(b - a)^2}{a} \). In order for this to be a perfect square, \( a \) must be a perfect square and \( a \mid (b - a)^2 \) so \( a \mid b^2 \implies \sqrt{a} \mid b \). Since \( 1 \leq a \leq 100 \), \( 1 \leq \sqrt{a} \leq 10 \) and the possible choices for \( b \) are \( 2\sqrt{a}, 3\sqrt{a}, \ldots, [100/\sqrt{a}]\sqrt{a} \) or a total of \([100/\sqrt{a}] - 1\) possibilities. Note that \( b \neq a \) since the difference must be positive. So our answer is:

\[
\sum_{i=1}^{10} \left\lfloor \frac{100}{i} \right\rfloor - 1 = 281
\]

*Author: Roy Zhao*
4. [4] There are real numbers $a, b, c, d$ such that for all $(x, y)$ satisfying $6y^2 = 2x^3 + 3x^2 + x$, if $x_1 = ax + b$ and $y_1 = cy + d$, then $y_1^2 = x_1^3 - 36x_1$. What is $a + b + c + d$?

Solution:

We have $6y^2 = 2x^3 + 3x^2 + x$. Make the following substitution: $y = 3y', x = 3x'$. This gives $y'^2 = x'^3 + \frac{1}{2}x'^2 + \frac{1}{18}x'$. Further substitute $x'' = x' + \frac{1}{6}$. This gives

\[
y'^2 = x''^3 + \frac{1}{36}x''^2
\]

\[
36^3y'^2 = 36^3x''^3 - 36^2x''
\]

\[
(6^3y')^2 = (36x'')^3 - 36(36x'').
\]

Finally, make the substitution $y_1 = 216y'$ and $x_1 = 36x''$. Thus $y_1^2 = x_1^3 - 36x_1$, and by substitution, $y_1 = 72y$ and $x_1 = 12x + 6$. So our answer is $12 + 6 + 72 + 0 = 90$.

EDIT: This problem did not specify that $a$ and $c$ should be non-zero. As such, any solution using $a = c = 0$ and $(b, d)$ a solution to $y_1^2 = x_1^3 - 36x_1$ was valid. This problem has been thrown out due to such a wide variety of possible answers.

Author: Heesu Hwang

5. [5] Find the sum of the non-repeated roots of the polynomial $P(x) = x^6 - 5x^5 - 4x^4 - 5x^3 + 8x^2 + 7x + 7$.

Solution:

Note that $P(x) = (x^6 - 5x^5 - 4x^4 - 5x^3 + x^2) + (7x^2 + 7x + 7) = x^2(x^4 - 5x^3 - 4x^2 - 5x + 1) + 7(x^2 + x + 1)$. We check that $(x^4 - 5x^3 - 4x^2 - 5x + 1) = (x^2 + x + 1)(x^2 - 6x + 1)$. Thus $P(x) = (x^4 + x + 1)(x^2 - 6x + 1) + 7(x^2 + x + 1)$. There are still no convenient roots of the quartic, thus we again use standard guess and check with quadratics of constant terms 7 and 1 to find $(x^2 + x + 1)(x^2 - 7x + 7) = x^4 - 6x^2 + x^2 + 7 = P(x) = x^2 + x + 1)^2(x^2 - 7x + 7)$. The non repeated roots are those of $(x^2 - 7x + 7)$, and thus by Vieta’s, the sum of the non repeated roots is $7$.

Author: Christopher Zhang

6. [6] Define the sequence $a_n$ as follows: $a_1 = 1$, $a_2 = 2015$, and $a_n = \frac{na_{n-1}^2}{a_{n-1} + na_{n-2}}$ for $n > 2$. What is the least $k$ such that $a_k < a_{k-1}$?

Solution:

The recursion is equivalent to $\frac{a_{n-1}}{a_n} = \frac{a_{n-2}}{a_{n-1}} + \frac{1}{n} = \frac{1}{2015} + \sum_{i=3}^{n} \frac{1}{i}$. The first $k$ for which $\frac{a_{k-1}}{a_k} > 1$ occurs when $k = 7$ by a simple computation of the sum.

Author: Bill Huang

7. [7] We define the function $f(x, y) = x^3 + (y - 4)x^2 + (y^2 - 4y + 4)x + (y^3 - 4y^2 + 4y)$. Then choose any distinct $a, b, c \in \mathbb{R}$ such that the following holds: $f(a, b) = f(b, c) = f(c, a)$. Over all such choices of $a, b, c$, what is the maximum value achieved by:

\[
\min(a^4 - 4a^3 + 4a^2, b^4 - 4b^3 + 4b^2, c^4 - 4c^3 + 4c^2)
\]

Solution:

Let $f(x) = x^4 - 4x^3 + 4x^2$ then the equalities become

\[
\frac{f(b) - f(a)}{b - a} = \frac{f(c) - f(b)}{c - b} = \frac{f(a) - f(c)}{a - c}
\]

So this implies that the points $(a, f(a)), (b, f(b)), (c, f(c))$ lie on the same line and the question
now becomes, over all such lines which intersect \( f(x) \) at least 3 times, what is the maximum \( y \) value of the 3rd highest intersection point.

We claim that the highest intersection points occurs when the line is horizontal and tangent to \( f(x) \) at \( x = 1, f(x) = 1 \). If we choose \( a, b \) such that \( f(a), f(b) > 1 \), then it is clear from looking at the graph of \( f(x) \) that the line between \((a, f(a)), (b, f(b))\) can only intersect \( f(x) \) two times. And there are three solutions to \( f(x) = 1 \) which means that 1 is achievable. Therefore 1 is our answer (one possibility \((a, b, c) = (1 - \sqrt{2}, 1, 1 + \sqrt{2})\)).

Author: Roy Zhao

8. [8] We define the ridiculous numbers recursively as follows:

(a) 1 is a ridiculous number.

(b) If \( a \) is a ridiculous number, then \( \sqrt{a} \) and \( 1 + \sqrt{a} \) are also ridiculous numbers.

A closed interval \( I \) is “boring” if

- \( I \) contains no ridiculous numbers, and
- There exists an interval \([b, c]\) containing \( I \) for which \( b \) and \( c \) are both ridiculous numbers.

The smallest non-negative \( l \) such that there does not exist a boring interval with length \( l \) can be represented in the form \( \frac{a + b\sqrt{c}}{d} \) where \( a, b, c, d \) are integers, \( \gcd(a, b, d) = 1 \), and no integer square greater than 1 divides \( c \). What is \( a + b + c + d \)?

Solution:

The smallest ridiculous number is 1. This is true because if \( a \geq 1 \), then \( \sqrt{a} \geq 1 \) and \( 1 + \sqrt{a} \geq 1 \).

The supremum of the ridiculous numbers (the smallest number that is greater than all ridiculous numbers) is \( \frac{3 + \sqrt{5}}{2} \). This is because the largest ridiculous number that is \( n \) recursive steps away from 1 is \( 1 + \sqrt{1 + \sqrt{1 + \ldots + \sqrt{1}}} \), where there are \( n \) square root signs. As \( n \) approaches infinity, the largest ridiculous numbers approach \( M = 1 + \sqrt{1 + \sqrt{1 + \sqrt{1 + \ldots}}} \). It can be seen that \( M = \frac{3 + \sqrt{5}}{2} \) by observing that \( M \) satisfies the identity \( M = 1 + \sqrt{M} \).

There are no ridiculous numbers in the interval \([1 + \frac{\sqrt{5}}{2}, 2]\): In the case that \( r \) is of the form \( 1 + \sqrt{a} \) for some ridiculous \( a \), then \( r \) must be at least 2, because \( a \) must be at least 1. In the case that \( r \) is of the form \( \sqrt{a} \), then \( r \) must be less than \( \sqrt{\frac{3 + \sqrt{5}}{2}} = \frac{1 + \sqrt{5}}{2} \). Since these are the only two cases, if \( I = [1 + \frac{\sqrt{5}}{2}, 2 - \epsilon] \) for some small \( \epsilon > 0 \), then \( I \) is a boring interval, so \( l \) must be at least \( 2 - \frac{1 + \sqrt{5}}{2} = \frac{3 - \sqrt{5}}{2} \).

Now we show that there does not exist a boring interval of length \( \frac{3 - \sqrt{5}}{2} \). Assume for the sake of contradiction that such an interval exists. It must be contained in \([1, \frac{1 + \sqrt{5}}{2}] \) or \([2, \frac{3 + \sqrt{5}}{2}] \). The first case is ruled out because \( \sqrt{\frac{5}{2}} \) and \( \sqrt{2} \) are ridiculous numbers. The second case is ruled out because \( 1 + \sqrt{\frac{5}{2}} \) and \( 1 + \sqrt{2} \) are ridiculous. Therefore \( l = \frac{3 - \sqrt{5}}{2} \) and our answer is \( 3 + (-1) + 5 + 2 = 9 \).

Author: Ben Edelman