Algebra B

1. [**3**] Roy is starting a baking company and decides that he will sell cupcakes. He sells n cupcakes for $(n+20)(n+15)$ cents. A man walks in and buys \$10.50 worth of cupcakes. Roy bakes cupcakes at a rate of 10 cupcakes an hour. How many minutes will it take Roy to complete the order?

   Solution:

   Solving for x, we have $(x+20)(x+15) = 1050 \implies x^2 + 35x - 750 = (x+50)(x-15) = 0$ and since $x \geq 0$, $x = 15$ cupcakes. So it will take Roy 1.5 hours or $\boxed{90}$ minutes.

   Author: Heesu Hwang

2. [**3**] Let f be a function which takes in $0, 1, 2$ and returns $0, 1,$ or 2. The values need not be distinct: for instance we could have $f(0) = 1, f(1) = 1, f(2) = 2$. How many such functions are there which satisfy:

   $$f(2) + f(f(0)) + f(f(f(1))) = 5$$

   Solution:

   Since f must return $0, 1,$ or 2, we know that of the three values $f(2), f(f(0)), f(f(f(1)))$, 2 of them must be 2 and one must be 1 because that is the only way to obtain 5.

   Case 1: $f(2) = 1$. Then $f(f(0)) = 2$ and we can verify that this means that $f(0) = 1$ and $f(1) = 2$. And then $f(f(f(1))) = 2$ as required. So this is 1 function.

   Case 2: $f(f(0)) = 1$. Then $f(2) = 2$. If $f(0) = 0$, then $f(f(0)) = 0$ so we must have $f(0) = 1$ and $f(1) = 1$. But then $f(f(f(1))) = 1$ and $f(2) + f(f(0)) + f(f(f(1))) = 4$ so there are no functions with $f(f(0)) = 1$.

   Case 3: $f(f(f(1))) = 1$. Then $f(2) = 2$ and $f(f(0)) = 2$. So $f(0)$ is 1 or 2. If $f(0) = 1$, then $f(1) = f(f(0)) = 2$ and $f(f(f(1))) = 2$ which isn't the case. So $f(0) = 2$ as well. Then $f(f(f(1))) = 1$ and this can only happen if $f(1) = 1$. So this is another such function.

   In total there are only $\boxed{2}$ such functions, namely $(f(0), f(1), f(2)) = (1, 2, 1), (2, 1, 2)$.

   Author: Roy Zhao

3. [**4**] Andrew and Blair are bored in class and decide to play a game. They pick a pair (a, b) with $1 \leq a, b \leq 100$. Andrew says the next number in the geometric series that begins with a, b and Blair says the next number in the arithmetic series that begins with a, b. For how many pairs (a, b) is Andrew's number minus Blair's number a positive perfect square?

   Solution:

   Andrew will say $\frac{b^2}{a}$ and Blair will say $2b - a$ and hence the difference will be $\frac{b^2 - 2ab + a^2}{a} = \frac{(b-a)^2}{a}$. In order for this to be a perfect square, a must be a perfect square and $a \mid (b-a)^2$ so $a \mid b^2 \implies \sqrt{a} \mid b$. Since $1 \leq a \leq 100$, $1 \leq \sqrt{a} \leq 10$ and the possible choices for b are $2\sqrt{a}, 3\sqrt{a}, \ldots, \lfloor 100/\sqrt{a} \rfloor \sqrt{a}$ or a total of $\lfloor 100/\sqrt{a} \rfloor - 1$ possibilities. Note that $b \neq a$ since the difference must be positive. So our answer is:

   $$\sum_{i=1}^{10} \left\lfloor \frac{100}{i} \right\rfloor - 1 = \boxed{281}$$

   Author: Roy Zhao

4. [**4**] There are real numbers a, b, c, d such that for all (x, y) satisfying $6y^2 = 2x^3 + 3x^2 + x$, if $x_1 = ax + b$ and $y_1 = cy + d$, then $y_1^2 = x_1^3 - 36x_1$. What is $a + b + c + d$?

   Solution:

   We have $6y^2 = 2x^3 + 3x^2 + x$. Make the following substitution: $y = 3y'$, $x = 3x'$. This gives $y'^2 = x'^3 + \frac{1}{2}x'^2 + \frac{1}{18}x'$. Further substitute $x'' = x' + \frac{1}{6}$. This gives

$$y'^2 = x''^3 - \frac{1}{36}x''$$
$$36^3y'^2 = 36^3x''^3 - 36^2x''$$
$$(6^3y')^2 = (36x'')^3 - 36(36x'').$$

   Finally, make the substitution $y_1 = 216y'$ and $x_1 = 36x''$. Thus $y_1^2 = x_1^3 - 36x_1$, and by substitution, $y_1 = 72y$ and $x_1 = 12x + 6$. So our answer is $12 + 6 + 72 + 0 = \boxed{90}$.

   EDIT: This problem did not specify that a and c should be non-zero. As such, any solution using $a = c = 0$ and (b, d) a solution to $y_1^2 = x_1^3 - 36x_1$ was valid. This problem has been thrown out due to such a wide variety of possible answers.

   Author: Heesu Hwang

5. [**5**] Find the sum of the non-repeated roots of the polynomial $P(x) = x^6 - 5x^5 - 4x^4 - 5x^3 + 8x^2 + 7x + 7$.

   Solution:

   Note that $P(x) = (x^6 - 5x^5 - 4x^4 - 5x^3 + x^2) + (7x^2 + 7x + 7) = x^2(x^4 - 5x^3 - 4x^2 - 5x + 1) + 7(x^2 + x + 1)$. We check that $(x^4 - 5x^3 - 4x^2 - 5x + 1) = (x^2 + x + 1)(x^2 - 6x + 1)$. Thus $P(x) = (x^2 + x + 1)(x^2(x^2 - 6x + 1) + 7) = (x^2 + x + 1)(x^4 - 6x^3 + x^2 + 7)$. There are still no convenient roots of the quartic, thus we again use standard guess and check with quadratics of constant terms 7 and 1 to find $(x^2 + x + 1)(x^2 - 7x + 7) = x^4 - 6x^2 + x^2 + 7 \implies P(x) = (x^2 + x + 1)^2(x^2 - 7x + 7)$. The non repeated roots are those of $(x^2 - 7x + 7)$, and thus by Vieta's, the sum of the non repeated roots is $\boxed{7}$.

   Author: Christopher Zhang

6. [**6**] Define the sequence a_i as follows: $a_1 = 1, a_2 = 2015$, and $a_n = \frac{na_{n-1}^2}{a_{n-1} + na_{n-2}}$ for $n > 2$. What is the least k such that $a_k < a_{k-1}$?

   Solution:

   The recursion is equivalent to $\frac{a_{n-1}}{a_n} = \frac{a_{n-2}}{a_{n-1}} + \frac{1}{n} = \frac{1}{2015} + \sum_{i=3}^{n} \frac{1}{i}$. The first k for which $\frac{a_{k-1}}{a_k} > 1$ occurs when $k = \boxed{7}$ by a simple computation of the sum.

   Author: Bill Huang

7. [**7**] We define the function $f(x, y) = x^3 + (y - 4)x^2 + (y^2 - 4y + 4)x + (y^3 - 4y^2 + 4y)$. Then choose any distinct $a, b, c \in \mathbb{R}$ such that the following holds: $f(a, b) = f(b, c) = f(c, a)$. Over all such choices of a, b, c, what is the maximum value achieved by:

$$\min(a^4 - 4a^3 + 4a^2, b^4 - 4b^3 + 4b^2, c^4 - 4c^3 + 4c^2)?$$

   Solution:

   Let $f(x) = x^4 - 4x^3 + 4x^2$ then the equalities become $\frac{f(b) - f(a)}{b - a} = \frac{f(c) - f(b)}{c - b} = \frac{f(a) - f(c)}{a - c}$. So this implies that the points $(a, f(a)), (b, f(b)), (c, f(c))$ lie on the same line and the question now becomes, over all such lines which intersect $f(x)$ at least 3 times, what is the maximum y value of the 3rd highest intersection point.

We claim that the highest intersection points occurs when the line is horizontal and tangent to $f(x)$ at $x = 1, f(x) = 1$. If we choose a, b such that $f(a), f(b) > 1$, then it is clear from looking at the graph of $f(x)$ that the line between $(a, f(a)), (b, f(b))$ can only intersect $f(x)$ two times. And there are three solutions to $f(x) = 1$ which means that 1 is achievable. Therefore $\boxed{1}$ is our answer (one possibility $(a, b, c) = (1 - \sqrt{2}, 1, 1 + \sqrt{2})$).

Author: Roy Zhao

8. [**8**] We define the ridiculous numbers recursively as follows:

    (a) 1 is a ridiculous number.

    (b) If a is a ridiculous number, then $\sqrt{a}$ and $1 + \sqrt{a}$ are also ridiculous numbers.

    A closed interval I is "boring" if

    - I contains no ridiculous numbers, and
    - There exists an interval $[b, c]$ containing I for which b and c are both ridiculous numbers.

    The smallest non-negative l such that there does not exist a boring interval with length l can be represented in the form $\dfrac{a + b\sqrt{c}}{d}$ where a, b, c, d are integers, $\gcd(a, b, d) = 1$, and no integer square greater than 1 divides c. What is $a + b + c + d$?

    Solution:

    The smallest ridiculous number is 1. This is true because if $a \geq 1$, then $\sqrt{a} \geq 1$ and $1+\sqrt{a} \geq 1$. The supremum of the ridiculous numbers (the smallest number that is greater than all ridiculous numbers) is $\frac{3+\sqrt{5}}{2}$. This is because the largest ridiculous number that is n recursive steps away from 1 is $1 + \sqrt{1 + \sqrt{1 + \dots \sqrt{1}}}$, where there are n square root signs. As n approaches infinity, the largest ridiculous numbers approach $M = 1 + \sqrt{1 + \sqrt{1 + \sqrt{1 + \dots}}}$. It can be seen that $M = \frac{3+\sqrt{5}}{2}$ by observing that M satisfies the identity $M = 1 + \sqrt{M}$.

    There are no ridiculous numbers in the interval $[\frac{1+\sqrt{5}}{2}, 2)$: In the case that r is of the form $1 + \sqrt{a}$ for some ridiculous a, then r must be at least 2, because a must be at least 1. In the case that r is of the form $\sqrt{a}$, then r must be less than $\sqrt{\frac{3+\sqrt{5}}{2}} = \frac{1+\sqrt{5}}{2}$. Since these are the only two cases, if $I = [\frac{1+\sqrt{5}}{2}, 2 - \epsilon]$ for some small $\epsilon > 0$, then I is a boring interval, so l must be at least $2 - \frac{1+\sqrt{5}}{2} = \frac{3-\sqrt{5}}{2}$.

    Now we show that there does not exist a boring interval of length $\frac{3-\sqrt{5}}{2}$. Assume for the sake of contradiction that such an interval exists. It must be contained in $[1, \frac{1+\sqrt{5}}{2}]$ or $[2, \frac{3+\sqrt{5}}{2}]$. The first case is ruled out because $\sqrt{\sqrt{2}}$ and $\sqrt{2}$ are ridiculous numbers. The second case is ruled out because $1 + \sqrt{\sqrt{2}}$ and $1 + \sqrt{2}$ are ridiculous. Therefore $l = \frac{3-\sqrt{5}}{2}$ and our answer is $3 + (-1) + 5 + 2 = \boxed{9}$.

    Author: Ben Edelman