PUMaC 2015

Combinatorics A Solutions

1. [**3**] A word is an ordered, non-empty sequence of letters, such as *word* or *wrod*. How many distinct 3-letter words can be made from a subset of the letters c, o, m, b, o, where each letter in the list is used no more than the number of times it appears?

   Solution: There are $\binom{3}{2} \cdot 3 = 9$ three-letter words with 2 o's (we choose the positions of the o's and then choose the third letter) and $4 \cdot 3 \cdot 2 = 24$ words without an o for a total of $\boxed{33}$ words.

   Author: Bill Huang

2. [**3**] Jonathan has a magical coin machine which takes coins in amounts of 7, 8, and 9. If he puts in 7 coins, he gets 3 coins back; if he puts in 8, he gets 11 back; and if he puts in 9, he gets 4 back. The coin machine does not allow two entries of the same amount to happen consecutively. Starting with 15 coins, what is the minimum number of entries he can make to end up with 4 coins?

   Solution: Jonathan can put in coins in the sequence 8, 9, 7, and 9. The number of coins he will have after each step is, respectively, 18, 13, 9, and 4. To arrive at this answer, work backwards from 4, listing the possible number of coins that would allow him to arrive at 4, and work up from there. Then, the minimum possible number of entries is $\boxed{4}$.

   Author: Jonathan Lu

3. [**4**] Princeton's Math Club recently bought a stock for \$2 and sold it for \$9 thirteen days later. Given that the stock either increases or decreases by \$1 every day and never reached \$0, in how many possible ways could the stock have changed during those thirteen days?

   Solution: We see that for this to happen, out of the 13 days, the stock price must increase 10 of those days and decrease the other 3 days. The number of arrangements of such days is the same as the number of ways to start at the origin and move to the lattice point $(3, 10)$ by only moving one unit to the right or up each step. There are $\binom{13}{3} = 286$ such ways to do this.

   However, we counted the case where the stock decreases to less than or equal to \$0, which is not allowed to happen. If the stock goes down the the first two days, there are $\binom{11}{1} = 11$ ways to distribute the last down day. Else, the only way the stock reaches \$0 is if the stock values fluctuates as $\{2, 1, 2, 1, 0\}$ or $\{2, 3, 2, 1, 0\}$, in which case for both, there is only one way to distribute the rest of the days. Thus there are 13 cases to subtract.

   Subtracting those cases, the total number of ways the stock could have changed is $\boxed{273}$.

   Author: Roy Zhao

4. [**4**] Andrew has 10 balls in a bag, each a different color. He randomly picks a ball from the bag 4 times, with replacement. The expected number of distinct colors among the balls he picks is $\frac{p}{q}$, where $\gcd(p, q) = 1$ and $p, q > 0$. What is $p + q$?

Solution: The probability that any particular one of the 10 colors is picked is $p = 1 - \left(\frac{9}{10}\right)^4 = \frac{3439}{10000}$. The expected contribution towards the total number of distinct colors picked by any particular color is then $p \cdot 1 + (1-p) \cdot 0 = p$, and by linearity of expectation, since we have 10 colors, the expected total number of distinct colors is $E = 10 \cdot p = \frac{3439}{1000}$, so $p = 3439$ and $q = 1000$ and $p + q = \boxed{4439}$.

Author: Roy Zhao

5. [**5**] Consider a random permutation of the set $\{1, 2, \ldots, 2015\}$. In other words, for each $1 \leq i \leq 2015$, i is sent to the element a_i where $a_i \in \{1, 2, \ldots, 2015\}$ and if $i \neq j$, then $a_i \neq a_j$. What is the expected number of ordered pairs (a_i, a_j) with $i - j > 155$ and $a_i - a_j > 266$?

Solution: First, observe that the total number of ordered pairs a_i, a_j satisfying $i - j > 155$ is equal to $(2015 - 156) + (2015 - 157) + \ldots + 1 = \binom{2015-155}{2} = 1728870$, where we count by casework on $j = 1, 2, \ldots, 1859$.

Since the permutation is random, the probability that any arbitrary ordered pair of elements a_i, a_j satisfy $a_i - a_j > n$ for some n is the same for any i, j. Furthermore, this probability is equal to the number of ordered pairs x, y satisfying $x - y > n$ and the total number of ordered pairs (x, y), as (a_i, a_j) is equally likely to be any ordered pair x, y (the former counted similarly as above). Thus, when $n = 266$ the probability is:

$$p = \frac{\binom{2015-266}{2}}{2015 \cdot 2014} = \frac{759}{2015}$$

By linearity of expectation, we have that the expected total number of ordered pairs a_i, a_j satisfying $i - j > 155$ that also satisfy $a_i - a_j > 266$ is:

$$E = 1728870 \cdot \frac{759}{2015} = \boxed{651222}$$

Authors: Roy Zhao, Bill Huang

6. [**6**] A number is *interesting* if it is a 6-digit integer that contains no zeros, its first 3 digits are strictly increasing, and its last 3 digits are non-increasing. What is the average of all interesting numbers?

Solution: We calculate the expected value of each digit, then use linearity of expectation to find the total expected value. Let $a_1, a_2, a_3, a_4, a_5, a_6$ denote the expected value of each digit and let the expected value be $a = \overline{a_1 a_2 a_3 a_4 a_5 a_6}$. By symmetry, we have $a_2 = a_5 = 5$. and $a_1 + a_3 = a_4 + a_6 = 10$.

If $a_1 = k$, then $a_2, a_3 \in \{k+1, \ldots, 9\}$ and (a_2, a_3) correspond to the number of ways $\binom{9-k}{2}$ to choose two numbers among $\{k+1, \ldots, 9\}$, so calculating and using the hockey stick identity:

$$a_1 = \frac{1 \cdot \binom{8}{2} + 2 \cdot \binom{7}{2} + \ldots + 7 \cdot \binom{2}{2}}{\binom{8}{2} + \binom{7}{2} + \ldots + \binom{2}{2}} = \frac{\binom{9}{3} + \binom{8}{3} + \ldots + \binom{3}{3}}{\binom{9}{3}} = \frac{\binom{10}{4}}{\binom{9}{3}} = \frac{10}{4} = \frac{5}{2}$$

If $a_6 = k$, then $a_5, a_4 \in \{k, \ldots, 9\}$ and (a_5, a_4) correspond to the number of ways $\binom{9-k+2}{2}$ to choose two numbers among $\{k, \ldots, 9\}$ allowing replacement, so calculating and using the hockey stick identity:

$$a_6 = \frac{1 \cdot \binom{10}{2} + 2 \cdot \binom{9}{2} + \ldots + 9 \cdot \binom{2}{2}}{\binom{10}{2} + \binom{9}{2} + \ldots + \binom{2}{2}} = \frac{\binom{11}{3} + \binom{10}{3} + \ldots + \binom{3}{3}}{\binom{11}{3}} = \frac{\binom{12}{4}}{\binom{11}{3}} = \frac{12}{4} = 3$$

Then it follows that $a = \overline{a_1 a_2 a_3 a_4 a_5 a_6} = 250000 + 50000 + 7500 + 700 + 50 + 3 = \boxed{308253}$.

Authors: Victor Zhou, Bill Huang

7. [**7**] Alice has an orange 3-by-3-by-3 cube, which is comprised of 27 distinguishable, 1-by-1-by-1 cubes. Each small cube was initially orange, but Alice painted 10 of the small cubes completely black. In how many ways could she have chosen 10 of these smaller cubes to paint black such that every one of the 27 3-by-1-by-1 sub-blocks of the 3-by-3-by-3 cube contains at least one small black cube?

Solution: Divide the 3-by-3-by-3 cube into 3 1-by-3-by-3 blocks. If 10 total smaller cubes are painted black, then two of these blocks must contain 3 black cubes and the third contains 4. Now, if a black does not have a diagonal of black cubes (allowing wrap-arounds), it must contain at least 4 cubes, so there are at least two blocks with diagonals and with exactly 3 cubes. We consider two cases.

<u>Case 1:</u> The diagonals of these two blocks are oriented in the same direction.

Clearly, the third block must contain a diagonal oriented in the same direction as well. The remaining black cube can be anywhere else in the block. There are $3 \cdot 6 \cdot 2 = 36$ ways to choose the first two blocks and their diagonals. There are $1 \cdot 6 = 6$ ways to choose black cubes in the remaining block. This gives a total of 216 colorings.

<u>Case 2:</u> They are oriented in opposite directions.

Then, the black cubes in the remaining block is determined (consider the projection of the blocks on top of one another; four squares are missing and the remaining block contains four black cubes). There are $3 \cdot 6 \cdot 3 = 54$ ways to choose the first to blocks and their diagonals. There is only 1 way to choose the black cubes in the remaining block. This gives a total of 54 colorings.

In total, then, there are $216 + 54 = \boxed{270}$ ways to choose 10 of the smaller cubes to paint black.

Author: Bill Huang

8. [**8**] In how many ways can 9 cells of a 6-by-6 grid be painted black such that no two black cells share a corner or an edge with each other?

Solution: Clearly, each 2-by-2 subgrid can only contain one black cell, as otherwise, two black cells in the same 2-by-2 subgrid must share the center corner. We then split the grid into 9 2-by-2 subgrids with corners $(2a, 2b), (2a+2, 2b), (2a, 2b+2), (2a+2, 2b+2)$ for $a, b \in \{0, 1, 2\}$. Thus, each subgrid must contain exactly one black cell.

The placement of each black cell in each 2-by-2 subgrid can be determined by two parameters: left or right and top or bottom. If subgrid (a, b) is right, then $(a+1, b)$ must be right also, and if (a, b) is left, then $(a-1, b)$ must be left also, and similarly for the top/bottom parameters. Then, looking at the first right subgrid in a row and first bottom subgrid in a column, there are $(3+1)^{2\cdot 3} = 4096$ potential placements of black cells.

However, not all these placements satisfy the conditions, as two black cells in diagonally-adjacent subgrids can share a corner. If the corner $(2, 2)$ is shared, then there are $(1 \cdot 2 \cdot 4) \cdot (1 \cdot 2 \cdot 4) = 8^2$ placements where the top left and bottom right cells of the corner are black and $(2 \cdot 1 \cdot 4) \cdot (2 \cdot 1 \cdot 4) = 8^2$ where the top right and bottom left cells of the corner are black, giving a total of 128 invalid placements. Similarly, counting invalid placements where the corners $(2, 4), (4, 2), (4, 4)$ are shared, we count a total of 512 invalid placements.

This overcounts a few invalid placements; namely, those where corners $(2, 4)$ and $(4, 2)$ or $(2, 2)$ and $(4, 4)$ are shared. For each of these pairs, there are 2 possible orientations (top left and bottom right v. top right and bottom left) and 2^2 ways to place the rest of the black cells, giving a total of 16 overcounted invalid placements.

By PIE, the total number of valid placements is then $4096 - 512 + 16 = \boxed{3600}$.

Author: Bill Huang