PUMaC 2015

Number Theory B

1. [**3**] What is the remainder when
$$\sum_{k=0}^{100} 10^k$$
is divided by 9?

2. [**3**] What is the 22nd positive integer n such that 22^n ends in a 2? (when written in base 10).

3. [**4**] What is the sum of all positive integers n such that $\text{lcm}(2n, n^2) = 14n - 24$?

4. [**4**] A circle with radius 1 and center $(0, 1)$ lies on the coordinate plane. Ariel stands at the origin and rolls a ball of paint at an angle of 35 degrees relative to the positive x-axis (counting degrees counterclockwise). The ball repeatedly bounces off the circle and leaves behind a trail of paint where it rolled. After the ball of paint returns to the origin, the paint has traced out a star with n points on the circle. What is n?

5. [**5**] Given that there are 24 primes between 3 and 100, inclusive, what is the number of ordered pairs (p, a) with p prime, $3 \le p < 100$, and $1 \le a < p$ such that $p \mid (a^{p-2} - a)$?

6. [**6**] What is the largest positive integer n less than $10,000$ such that in base 4, n and $3n$ have the same number of digits; in base 8, n and $7n$ have the same number of digits; and in base 16, n and $15n$ have the same number of digits? Express your answer in base 10.

7. [**7**] What is the smallest positive integer n such that $20 \equiv n^{15} \pmod{29}$?

8. [**8**] Given a positive integer k, let $f(k)$ be the sum of the k-th powers of the primitive roots of 73. For how many positive integers $k < 2015$ is $f(k)$ divisible by 73?

   Note: A primitive root r of a prime p is an integer $1 \le r < p$ such that the smallest positive integer k such that $r^k \equiv 1 \pmod{p}$ is $k = p - 1$.