PUMaC 2015

Number Theory B

1. [**3**] What is the remainder when

$$\sum_{k=0}^{100} 10^k$$

is divided by 9?

Solution:

The sum of the digits of the expression is 101, so by the rule for remainder modulo 9, the answer is $101 \pmod 9$, or $\boxed{2}$.

Author: Eric Neyman

2. [**3**] What is the 22nd positive integer n such that 22^n ends in a 2? (when written in base 10).

Solution:

The last digits of the powers of 22 cycle $2, 4, 8, 6, 2, 4, 8, 6, \ldots$. The answer is thus the 22nd term of the arithmetic sequence $1, 5, 9, \ldots,$ which is $\boxed{85}$.

Author: Eric Neyman

3. [**4**] What is the sum of all positive integers n such that $\text{lcm}(2n, n^2) = 14n - 24$?

Solution:

$\text{lcm}(2n, n^2)$ is n^2 when n is even and $2n^2$ when n is odd. Solving the equation $n^2 = 14n - 24$, we get $n = 2$ and $n = 12$, both of which are even and thus yield solutions. Solving $2n^2 = 14n - 24$, we get $n = 3$ and $n = 4$, of which only $n = 3$ works, because it is odd. Thus, our answer is $2 + 3 + 12 = \boxed{17}$.

Author: Eric Neyman

4. [**4**] A circle with radius 1 and center $(0, 1)$ lies on the coordinate plane. Ariel stands at the origin and rolls a ball of paint at an angle of 35 degrees relative to the positive x-axis (counting degrees counterclockwise). The ball repeatedly bounces off the circle and leaves behind a trail of paint where it rolled. After the ball of paint returns to the origin, the paint has traced out a star with n points on the circle. What is n?

Solution: The ball of paint bounces off the circle such that the angle between two consecutive bounces that is centered at the origin is $2\dot{3}5 = 70$ degree as usual. Therefore, the paint keeps bouncing until it hits the first multiple of $360°$ that is also a multiple of $70°$. Therefore, the number of unique intersections with the circle is

$$\frac{\text{lcm}(360, 70)}{70} = \frac{360 \cdot 7}{10 \cdot 7} = \boxed{36}.$$

Author: Heesu Hwang

5. [**5**] Given that there are 24 primes between 3 and 100, inclusive, what is the number of ordered pairs (p, a) with p prime, $3 \le p < 100$, and $1 \le a < p$ such that $p \mid (a^{p-2} - a)$?

Solution:

By Fermat's Little Theorem, $a^{p-1} \equiv 1 \pmod p$. This means that $a^{p-2} \equiv a^{-1} \pmod p$, so

$$0 \equiv a^{p-2} - a \equiv a^{-1} - a \pmod p \implies a^2 \equiv 1 \pmod p.$$

Clearly the only solutions are $a = 1, p-1$, so for each prime p there are exactly two solutions of the form (p, a). Hence there are $\boxed{48}$ solutions total.

Author: Steven Kwon

6. [**6**] What is the largest positive integer n less than 10,000 such that in base 4, n and $3n$ have the same number of digits; in base 8, n and $7n$ have the same number of digits; and in base 16, n and $15n$ have the same number of digits? Express your answer in base 10.

Solution:

The conditions are basically stating that in each of the three bases, n must appear "alphabetically" before $1111\ldots$. Let k_4, k_8, and k_{16} be the largest positive integers such that $4^{k_4} < n$, $8^{k_8} < n$, and $16^{k_{16}} < n$. Then $n < 2 \cdot 4^{k_4}$, $n < 2 \cdot 8^{k_8}$, and $n < 2 \cdot 16^{k_{16}}$. In other words, $n < 2^{2k_4+1}$, $n < 2^{3k_8+1}$, and $n < 2^{4k_{16}+1}$, but $n > 2^{2k_4}$, $n > 2^{3k_8}$. and $n > 2^{4k_{16}}$. It follows that the greatest power of 2 that is less than n must be 2^{12k} for some integer k (because 12 is the least common multiple of 2, 3, and 4). The largest such k can be 1 if n is to be less than 10,000. From here it is easy to see that the largest possible value of n is the smallest of 1111_{16}, 11111_8, and 1111111_4, and that is $1111_{16} = \boxed{4369}$.

Author: Eric Neyman

7. [**7**] What is the smallest positive integer n such that $20 \equiv n^{15} \pmod{29}$?

Solution:

By Fermat's Little Theorem, $a^{28} \equiv 1 \pmod{29}$ for all positive integers a that are not multiples of 29. It follows that $a^{14} \equiv \pm 1 \pmod{29}$, so $a^{15} \equiv \pm a \pmod{29}$ for all such a. Thus, if $a^{15} \equiv 20 \pmod{29}$, then $\pm a \equiv 20 \pmod{29}$. We know that $9^{14} = 3^{28} \equiv 1 \pmod{29}$, so $9^{15} \equiv 9 \pmod{29}$. Next we try $a = 20$, and we find that $20^{14} \equiv 49^{14} \equiv 7^{28} \equiv 1 \pmod{29}$, and so $20^{15} \equiv 20 \pmod{29}$. Therefore, the smallest such a is $\boxed{20}$.

Author: Heesu Hwang

8. [**8**] Given a positive integer k, let $f(k)$ be the sum of the k-th powers of the primitive roots of 73. For how many positive integers $k < 2015$ is $f(k)$ divisible by 73?

Note: A primitive root r of a prime p is an integer $1 \le r < p$ such that the smallest positive integer k such that $r^k \equiv 1 \pmod p$ is $k = p - 1$.

Solution:

Denote S_d^k to be the sum of the k-th powers of the residues of order d modulo $p = 73$. Now, for any primitive root r of p, we have:

$$\text{ord}_p(r^k) = \frac{\text{ord}_p(r)}{\gcd(k, p-1)} = \frac{p-1}{\gcd(k, p-1)} = d_k$$

Furthermore, the number of residues of p of order d is:

$$n_d = \phi(d)$$

Now, since d_k is independent of the primitive root r, S_1^k is equal to a sum of n_{p-1} residues of order d_k. More specifically, due to symmetry within the group $\mathbb{Z}/p\mathbb{Z}$, each residue or order d_k is represented in the sum the same number of times. Then, we have:

$$S_1^k = \frac{n_{p-1}}{n_{d_k}} \cdot S_{d_k} = \frac{\phi(p-1)}{\phi\left(\frac{p-1}{\gcd(k,p-1)}\right)} \cdot S_{d_k} = \phi\big(\gcd(k,p-1)\big) \cdot S_{d_k}$$

Clearly, $p \nmid \phi\big(\gcd(k,p-1)\big) < p$, so $p \mid S_{d_k}$. We now present a lemma:

Lemma 1: For some $d \mid p-1$, $S_d \equiv 0 \pmod{p}$ iff d is divisible by a perfect square. Furthermore, if $d = q_1 q_2 \ldots q_m$ is square-free, then we have that $S_d \equiv (-1)^m$.

Proof: We prove by induction. For $d = 1$, it is clear that $S_1 = 1 = (-1)^0$. Now, assume this is true for all $d' < d$. Let x be a residue of order d modulo p. Clearly, $p \mid x^{d-1} + x^{d-2} + \ldots + x + 1$, as $p \mid x^d - 1$ by assumption and $p \nmid x - 1$. But, we have:

$$x^{d-1} + x^{d-2} + \ldots + x + 1 = \sum_{d' \mid d} S_{d'} \equiv 0 \pmod{p}$$

If d is square-free, we write $d = q_1 q_2 \ldots q_m$. Then, we have:

$$S_d \equiv -\sum_{d' \mid d, d' \neq d} S_{d'} \equiv (-1)\left(\sum_{i=0}^{m-1} (-1)^i \binom{m}{i}\right) \equiv (-1)\left(-(-1)^m \binom{m}{m}\right) \equiv (-1)^m \pmod{p}$$

Otherwise, we write $d = q_1^{\alpha_1} q_2^{\alpha_2} \ldots q_m^{\alpha_m}$ and $e = q_1 q_2 \ldots q_m < d$. Then, we have:

$$S_d \equiv -\sum_{d' \mid d, d' \neq d} S_{d'} \equiv -\sum_{e' \mid e} S_{e'} \equiv (-1)\left(\sum_{i=0}^{m} (-1)^i \binom{m}{i}\right) \equiv 0 \pmod{p}$$

As desired. □

This means that k satisfies the condition iff d_k is divisible by a perfect square. Since $p - 1 = 72 = 2^3 \cdot 3^2$, d_k is square-free iff $2^2 \cdot 3 = 12 \mid k$, so k satisfies the condition iff $12 \nmid k$. There are $2014 - \lfloor \frac{2014}{12} \rfloor = \boxed{1847}$ such k.

Author: Bill Huang