PUMaC 2015

Individual Finals B

1. Alice places down n bishops on a 2015×2015 chessboard such that no two bishops are attacking each other. (Bishops attack each other if they are on a diagonal.) Her friend Bob notices that he is not able to place down a larger number of bishops such that any two still cannot attack one another. Find, with proof, n.

   Solution:

   We claim that the maximum number of bishops that can be placed in this fashion on an n-by-n chessboard is $2(n-1)$ when $n \leq 2$. First, notice that in any unoccupied squares, no more than two bishops can attack that square, or else by pigeonhole, two bishops can attack one another. Furthermore, a corner square can be attacked by at most one bishop, and there will be at most two unoccupied corner squares. Thus, if there are k bishops, the number of possible attacks is at most $2(n^2-k)-2$

   Now, each bishop can attack at least $n-1$ squares, with equality holding iff the bishop lies on the outer boundary of the chessboard. Thus, if there are k bishops, then we have the inequality:

   $$\begin{aligned} k(n-1) &\leq 2(n^2-k)-2 \\ k &\leq \frac{2n^2-2}{n+1} \\ &= 2(n-1) \end{aligned}$$

   As desired. Now, note that we may put bishops on $(1,i)$ with $i = 1, 2, \ldots n$ and (n, j) with $j = 2, 3 \ldots n-1$. There are $2n-2$ bishops and no two of them can attack each other. So the answer is $2 \cdot 2015 - 2 = \boxed{4028}$.

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2. On a circle ω_1, four points A, B, C, D lie in that order. Prove that $CD^2 = AC \cdot BC + AD \cdot BD$ if and only if at least one of C and D is the midpoint of arc AB.

   Solution:

   Without loss of generality, let the diameter of ω_1 be $\frac{1}{2}$. Notice that the area of quadrilateral $ABCD$ can be calculated in two ways: one as the sum of the area of $\triangle ABC$ and the area of $\triangle ABD$, which is $\frac{1}{2}AC \cdot BC \cdot sin\angle ACB + \frac{1}{2}AD \cdot BD \cdot sin\angle ADB$. The other way is to notice that the area is just $\frac{1}{2}AB \cdot CD \cdot sin\angle APC$ where P is the intersection of AB and CD. So we have:

   $$\frac{1}{2}AC \cdot BC \cdot sin\angle ACB + \frac{1}{2}AD \cdot BD \cdot sin\angle ADB = \frac{1}{2}AB \cdot CD \cdot sin\angle APC$$

   In a circle with diameter $\frac{1}{2}$, $\frac{1}{2}sin\angle ACB = \frac{1}{2}sin\angle ADB = AB$. Therefore $AC \cdot BC + AD \cdot BD = CD \cdot \frac{1}{2}sin\angle APC$. So $CD^2 = AC \cdot BC + AD \cdot BD$ iff $CD = \frac{1}{2}sin\angle APC$ iff $\angle CAD = \angle APC$ or $\angle CBD = \angle APC$.

   Since $\angle APC = \angle ADC + \angle BAD$, the previous statement is equivalent to $AC = BC$ or $AD = BD$, as desired.

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3. For an odd prime number p, let S denote the following sum taken modulo p:

   $$S \equiv \frac{1}{1 \cdot 2} + \frac{1}{3 \cdot 4} + \ldots + \frac{1}{(p-2) \cdot (p-1)} \equiv \sum_{i=1}^{\frac{p-1}{2}} \frac{1}{(2i-1) \cdot 2i} \pmod{p}.$$

Prove that $p^2|2^p-2$ if and only if $S \equiv 0 \pmod p$.

Solution:

Notice that the binomial coefficients satisfy the following basic properties:

$$\binom{p}{n} = \frac{p}{n}\binom{p-1}{n-1}$$

$$\binom{p-1}{n-1} = \frac{(p-1)(p-2)\dots(p-n+1)}{1\cdot 2\cdot 3\dots(n-1)} \equiv \frac{(-1)(-2)\dots(-n+1)}{1\cdot 2\dots(n-1)} = (-1)^{n-1} \mod p$$

Therefore we know that

$$S \equiv \sum_{i=1}^{\frac{p-1}{2}} \frac{1}{(2i-1)\cdot 2i} = \sum_{i=1}^{p-1} \frac{(-1)^{i-1}}{i} = \sum_{i=1}^{p-1} \frac{(-1)^{i-1}\binom{p}{i}}{p\cdot\binom{p-1}{i-1}} = \sum_{i=1}^{p-1} \frac{\binom{p}{i}}{p}\cdot\frac{(-1)^{p-1}}{\binom{p-1}{i-1}} \equiv \sum_{i=1}^{p-1} \frac{\binom{p}{i}}{p} = \frac{2^p-2}{p} \mod p$$

where $\frac{\binom{p}{i}}{p}$ is always viewed as an integer rather than a division in mod p.

Therefore, $p^2|2^p-2$ if and only if $S = 0 \mod p$.

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