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Algebra A

1. Let $a_1 = 20$, $a_2 = 16$, and for $k \geq 3$, let $a_k = \sqrt[3]{k - a_{k-1}^3 - a_{k-2}^3}$. Compute $a_1^3 + a_2^3 + \cdots + a_{10}^3$.

2. Let $f(x) = 15x - 2016$. If $f(f(f(f(f(x))))) = f(x)$, find the sum of all possible values of x.

3. For positive real numbers x and y, let $f(x, y) = x^{\log_2 y}$. The sum of the solutions to the equation
$$4096 f(f(x, x), x) = x^{13}$$
can be written in simplest form as $\frac{m}{n}$. Compute $m + n$.

4. Suppose that P is a polynomial with integer coefficients such that $P(1) = 2$, $P(2) = 3$ and $P(3) = 2016$. If N is the smallest possible positive value of $P(2016)$, find the remainder when N is divided by 2016.

5. Define a sequence a_i as follows: $a_1 = 181$ and for $i \geq 2$, $a_i = a_{i-1}^2 - 1$ if a_{i-1} is odd and $a_i = a_{i-1}/2$ if a_{i-1} is even. Find the least i such that $a_i = 0$.

6. Let $[a, b] = ab - a - b$. Shaq sees the numbers $2, 3, \ldots, 101$ written on a blackboard. Let V be the largest number that Shaq can obtain by repeatedly choosing two numbers a, b on the board and replacing them with $[a, b]$ until there is only one number left. Suppose N is the integer with $N!$ nearest to V. Find the nearest integer to $10^6 \cdot \frac{|V - N!|}{N!}$.

7. Let S_P be the set of all polynomials P with complex coefficients, such that $P(x^2) = P(x)P(x-1)$ for all complex numbers x. Suppose P_0 is the polynomial in S_P of maximal degree such that $P_0(1) \mid 2016$. Find $P_0(10)$.

8. Define the function $f : \mathbb{R} \backslash \{-1, 1\} \to \mathbb{R}$ to be
$$f(x) = \sum_{a,b=0}^{\infty} \frac{x^{2^a 3^b}}{1 - x^{2^{a+1} 3^{b+1}}}.$$
Suppose that $f(y) - f(\frac{1}{y}) = 2016$. Then y can be written in simplest form as $\frac{p}{q}$. Find $p + q$. ($\mathbb{R} \backslash \{-1, 1\}$ refers to the set of all real numbers excluding -1 and 1.)