Algebra A Solutions

1. We have $a_2^3 + a_3^3 + a_4^3 = 4$, $a_5^3 + a_6^3 + a_7^3 = 7$, $a_8^3 + a_9^3 + a_{10}^3 = 10$, and $a_1^3 = 8000$. Thus, the answer is $\boxed{8021}$.

   Problem written by Eric Neyman.

2. Note that $f(f(f(f(f(x))))) = f(x)$ is a linear equation in x and thus has one solution. If $f(x) = x$ then clearly this equation is satisfied. Thus the solution is the solution to the equation $x = 15x - 2016$, which is $\boxed{144}$.

   Problem written by Eric Neyman.

3. We have

$$\begin{aligned}
4096 f(f(x,x),x) &= x^{13} \\
4096\left(x^{\log_2 x}\right)^{\log_2 x} &= x^{13} \\
4096 x^{(\log_2 x)^2} &= x^{13} \\
4096 x^{(\log_2 x)^2 - 13} &= 1 \\
2^{(\log_2 x)((\log_2 x)^2 - 13) + 12} &= 2^0 \\
(\log_2 x)((\log_2 x)^2 - 13) + 12 &= 0 \\
(\log_2 x)^3 - 13\log_2 x + 12 &= 0 \\
(\log_2 x - 1)(\log_2 x - 3)(\log_2 x + 4) &= 0,
\end{aligned}$$

   so x can be any of 2, 8, and $\frac{1}{16}$. Thus, the sum of all possible values of x is $\frac{161}{16}$, and so our answer is $161 + 16 = \boxed{177}$.

   Problem written by Eric Neyman.

4. Using the well known result that $x_1 - x_2 \mid P(x_1) - P(x_2)$ we get that $N \equiv 2 \pmod{2015}$, $N \equiv 3 \pmod{2014}$ and $N \equiv 3 \pmod{2013}$. Solving these equations gives $N \equiv 3 + 1007 \times 2013 \times 2014 \pmod{2013 \times 2014 \times 2015}$. Thus $N = 3 + 1007 \times 2013 \times 2014$ and $N \equiv 3 + 1007 \times (-3) \times (-2) \equiv 2013 \pmod{2016}$. The polynomial $P(x) = (x-1)((x-2)(1006)+1)+2$ satisfies all given conditions, so $\boxed{2013}$ is the final answer.

   Problem written by Mel Shu.

5. By a computation, $a_5 = 2^{12} - 1$. If $a_i = 2^k - 1$, then $a_{i+1} = (2^{k-1} - 1)2^{k+1}$, so $a_{i+k+2} = 2^{k-1} - 1$. Eventually we get $a_{105} = 0$. Thus the answer is $\boxed{105}$.

   Problem written by Zhuo Qun Song.

6. We can write $[a,b] = (a-1)(b-1) - 1$. Since $[[a,b],[c,d]] \le [[[a,b],c],d]$ if $a > b > c > d$, we can maximize V by finding

$$[[\cdots[[101,100],99],\cdots],2] = \prod_{k=2}^{101}(k-1) - 2\sum_{j=2}^{99}\prod_{k=2}^{j}(k-1) - (2-1) - 1.$$

   This is perhaps most easily seen by evaluating

$$[[[a,b],c],d] = (a-1)(b-1)(c-1)(d-1) - 2(c-1)(d-1) - 2(d-1) - 1$$

and generalizing from there; the general form is

$$[[\cdots[[a_n,a_{n-1}],a_1],\cdots],2]=\prod_{k=1}^{n}(a_k-1)-2\sum_{j=1}^{n-2}\prod_{k=1}^{j}(a_k-1)-(a_1-1)-1.$$

Clearly the first term, 100!, dominates, so $N=100$. So, the desired value becomes $2\cdot 10^6(\frac{1}{100\cdot 99}+\frac{1}{100\cdot 99\cdot 98}+\frac{1}{100\cdot 99\cdot 98\cdot 97}+\cdots)$ as the remaining terms in the sum are far too small to change our value. In fact, $100\cdot 99\cdot 98\cdot 97>>2\cdot 10^6$, so we actually wish to find $2\cdot 10^4(\frac{1}{99}+\frac{1}{99\cdot 98})$. This is just a smidge over $\boxed{204}$.

Problem written by Zachary Stier.

7. We first determine all solutions to the equation $P(x^2)=P(x)P(x-1)$. Let $P(x)=\prod(x-\alpha_i)$ where α_i are the roots of P (including multiplicity). Then $\prod(x^2-\alpha_i)=\prod(x-\alpha_i)\prod(x-(\alpha_i+1))$. Thus, considering the sets of roots of both sides of this equation, we get $\{\pm\sqrt{\alpha_i}\}=\{\alpha_i\}\cup\{\alpha_i+1\}$. Now we note that if $\max\left|\sqrt{\alpha_i}\right|=M>1$ then $\max|\alpha_i|=M^2>M$, contradiction. Similarly, if $\min\left|\sqrt{\alpha_i}\right|=N<1$ then $\min|\alpha_i|=N^2<N$, contradiction. (Unless $N=0$. But this is impossible, since the multiplicity of zero as a root on both sides of the equation cannot be equal.) Therefore all roots α_i must satisfy $|\alpha_i|=|\alpha_i+1|=1$, so $\alpha_i=\pm\left(\frac{-1+\sqrt{3}i}{2}\right)$. Now counting the distinct roots, the two different roots must have the same multiplicity, so the solutions are $P(x)=(x^2+x+1)^n$ for some positive integer n, or $P(x)=0$ or $P(x)=1$ (constant solutions). It follows that $P_0(1)=3^n$, so $P_0(x)=(x^2+x+1)^2$. Therefore $P_0(10)=111^2=\boxed{12321}$.

   Problem written by Mel Shu.

8. For $|x|<1$, we have:

$$\begin{aligned}
f(x)-f(x^{-1})&=\sum_{a,b\geq 0}\frac{x^{2^a3^b}}{1-x^{2^{a+1}3^{b+1}}}+\frac{x^{-2^a3^b}}{x^{-2^{a+1}3^{b+1}}-1}\\
&=\sum_{a\geq 0}\sum_{b\geq 0}\frac{x^{2^a3^b}+x^{5\cdot 2^a3^b}}{1-x^{2^{a+1}3^{b+1}}}\\
&=\sum_{a\geq 0}\sum_{u\geq 0}\left(x^{2^a}\right)^{2u+1}\\
&=\sum_{v>0}x^v\\
&=\frac{x}{1-x}
\end{aligned}$$

   Similarly, for $|x|>1$, we have:

$$f(x)=f(x^{-1})=\frac{1}{1-x}.$$

   Since the range of $\frac{1}{1-x}$ over the domain $|x|>1$ is $(-\infty,0)\cup(0,\frac{1}{2})$, it follows that $|y|<1$. Setting $\frac{y}{1-y}=2016$ yields $y=\frac{2016}{2017}$, so $p+q=\boxed{4033}$.

   Problem written by Bill Huang.

If you believe that any of these answers is incorrect, or that a problem had multiple reasonable interpretations or was incorrectly stated, you may appeal at `tinyurl.com/pumacappeals`. All appeals must be in by 1 PM to be considered.