Combinatorics A

1. Chitoge is painting a cube; she can paint each face either black or white, but she wants no vertex of the cube to be touching three faces of the same color. In how many ways can Chitoge paint the cube? Two paintings of a cube are considered to be the same if you can rotate one cube so that it looks like the other cube.

2. 32 teams, ranked 1 through 32, enter a basketball tournament that works as follows: the teams are randomly paired and in each pair, the team that loses is out of the competition. The remaining 16 teams are randomly paired, and so on, until there is a winner. A higher-ranked team always wins against a lower-ranked team. If the probability that the team ranked 3 (the third-best team) is one of the last four teams remaining can be written in simplest form as $\frac{m}{n}$, compute $m + n$.

3. Alice, Bob, Charlie, Diana, Emma, and Fred sit in a circle, in that order, and each roll a six-sided die. Each person looks at his or her own roll, and also looks at the roll of either the person to the right or to the left, deciding at random. Then, at the same time, Alice, Bob, Charlie, Diana, Emma and Fred each state the expected sum of the dice rolls based on the information they have. All six people say different numbers; in particular, Alice, Bob, Charlie, and Diana say 19, 22, 21, and 23, respectively. Compute the product of the dice rolls.

4. A knight is placed at the origin of the Cartesian plane. Each turn, the knight moves in an chess L-shape (2 units parallel to one axis and 1 unit parallel to the other) to one of eight possible locations, chosen at random. After 2016 such turns, what is the expected value of the square of the distance of the knight from the origin?

5. Let $a_1, a_2, a_3, \ldots$ be an infinite sequence where for all positive integers i, a_i is chosen to be a random positive integer between 1 and 2016, inclusive. Let S be the set of all positive integers k such that for all positive integers $j < k$, $a_j \neq a_k$. (So $1 \in S$; $2 \in S$ if and only if $a_1 \neq a_2$; $3 \in S$ if and only if $a_1 \neq a_3$ and $a_2 \neq a_3$; and so on.) In simplest form, let $\frac{p}{q}$ be the expected number of positive integers m such that m and $m + 1$ are in S. Compute pq.

6. The George Washington Bridge is 2016 meters long. Sally is standing on the George Washington Bridge, 1010 meters from its left end. Each step, she either moves 1 meter to the left or 1 meter to the right, each with probability $\frac{1}{2}$. What is the expected number of steps she will take to reach an end of the bridge?

7. The Dinky is a train connecting Princeton to the outside world. It runs on an odd schedule: the train arrives once every one-hour block at some uniformly random time (once at a random time between 9am and 10am, once at a random time between 10am and 11am, and so on). One day, Emilia arrives at the station, at some uniformly random time, and does not know the time. She expects to wait for y minutes for the next train to arrive. After waiting for an hour, a train has still not come. She now expects to wait for z minutes. Find yz.

8. Katie Ledecky and Michael Phelps each participate in 7 swimming events in the Olympics (and there is no event that they both participate in). Ledecky receives g_L gold, s_L silver, and b_L bronze medals, and Phelps receives g_P gold, s_P silver, and b_P bronze medals. Ledecky notices that she performed objectively better than Phelps: for all positive real numbers $w_b < w_s < w_g$, we have
$$w_g g_L + w_s s_L + w_b b_L > w_g g_P + w_s s_P + w_b b_P.$$
Compute the number of possible 6-tuples $(g_L, s_L, b_L, g_P, s_P, b_P)$.