Combinatorics A Solutions

1. Pick a vertex of the cube. Suppose two faces that meet at that vertex are black and the other is white. The face opposite the white face is also white. Among the other two faces, not both are white. We thus get two possibilities: either one face of the other two is black, in which case the black faces form a "strip" of length 3, as do the white faces, or we have two opposite white faces and the other faces are black.

   The other possibility is that the two faces that meet at that vertex are white and the other is black. This gives the "strip" case again, as well as the case where we have two opposite black faces and the other faces are white. Thus, Chitoge can paint the cube in $\boxed{3}$ ways.

   Problem written by Bill Huang.

2. This is the same as putting the teams in a bracket-style tournament at random. The probability that the teams ranked 1 and 2 are not in the same quarter of the draw as the team ranked 3 is the relevant probability, and it is $\frac{24\cdot 23}{31\cdot 30} = \frac{92}{155}$, so the answer is $92 + 155 = \boxed{247}$.

   Problem written by Eric Neyman.

3. The sum of the two rolls each person sees is what they say minus 14 (the expected sum of the rolls they don't see). Since the stated numbers are all different, the sum of the two rolls each person sees is a different number, which means that no two people look at each other's dice, so everyone looks in the same direction. Assume that Alice looks at Bob, Bob looks at Charlie, and so on. (The other case is identical.) The sum of the two rolls that Alice, Bob, Charlie, and Diana see, respectively, is 5, 8, 7, and 9. If we let a through f be the rolls of Alice through Fred, this means that $a + b = 5$, $b + c = 8$, $c + d = 7$, and $d + e = 9$. Adding the second and fourth equations and subtracting the first and third gives $e - a = 5$, meaning that $e = 6$ and $a = 1$. Thus, $b = 4$, $c = 4$, and $d = 3$. It remains to determine f, and note that $e + f$ and $f + a$ do not belong to the set $\{5, 7, 8, 9\}$, for this would violate the condition that everyone says a different number. This rules out everything but $f = 5$. Thus, we have

   $$abcdef = 1 \cdot 4 \cdot 4 \cdot 3 \cdot 6 \cdot 5 = \boxed{1440}.$$

   Problem written by Eric Neyman.

4. Suppose the knight is at (a, b) before a turn. Consider the two possible moves $(-2, -1)$ and $(+2, +1)$. We have that $\frac{(a-2)^2+(b-1)^2+(a+2)^2+(b+1)^2}{2} = a^2+b^2+5$. The six other possible moves can be paired up similarly. Summing all possibilities, the expected value of $a^2 + b^2$ increases by 5 each turn, so after 2016 such turns, the answer is $2016 \cdot 5 = \boxed{10080}$

   Problem written by Mel Shu.

5. Let s_k be the kth element of S (ordered by size). k elements of $\{1, \ldots, 2016\}$ appear among $\{a_1, a_2, \ldots, a_{s_k}\}$, so the probability that a_{s_k+1} has not appeared among $\{a_1, a_2, \ldots, a_{s_k}\}$ is $\frac{2016-k}{2016}$, and this is the probability that $s_k + 1$ is also in S. Thus, by linearity of expectation, we have

   $$\frac{p}{q} = \frac{2015}{2016} + \frac{2014}{2016} + \cdots + \frac{1}{2016} = \frac{2015 \cdot 2016}{2 \cdot 2016} = \frac{2015}{2},$$

   so $pq = \boxed{4030}$.

   Problem written by Eric Neyman.

6. Note that the problem is symmetric in that if Sally stands at meter n or $2016-n$, the expected value of the number of steps to get off from those points are equal. Let $E[n]$ be the expected value of the number of steps Sally will take starting at meter n (or $2016-n$).

   Note that from meter 1008, Sally can either walk to meter 1007 or 1009. Thus

$$E[1008] = 1/2(E[1007]+1) + 1/2(E[1007]+1) = E[1007]+1.$$

   We can create further recursive equations in this form. For example,

$$\begin{aligned} E[1007] &= 1/2(E[1006]+1) + 1/2(E[1008]+1) \\ &= 1/2(E[1006]+1) + 1/2(E[1007]+1+1) \\ \implies &\ E[1007] = E[1006]+3. \end{aligned}$$

   We leave the proof by induction to the reader that for all $n \leq 1008$, $E[n] = E[n-1] + (1 + 2(1008-n))$. This further leads to the fact that $E[1] = 2015$.

   Note that we are asked to compute $E[1010] = E[1006]$. Thus by addition of these recursions, we have

$$E[1006] = 2015 + 2013 + 2011 + \cdots + 7 + 5 = 1008^2 - 1 - 3 = \boxed{1016060}.$$

   Problem written by Heesu Hwang.

7. Say that Emilia arrives at the station at hour x, $0 < x < 1$. (for example, if she arrives at 8:30, then $x = \frac{1}{2}$). Then, the probability that she misses both trains that could possibly arrive in the next hour is $x(1-x)$. The expected amount of time she then waits for the second train is $\frac{1}{2}(1-x)$.

   We can picture the weighted amount of time (over probability) as the tetrahedra $(0,0,0)$, $(1,0,0)$, $\left(0,0,\frac{1}{2}\right)$, $\left(0,1,\frac{1}{2}\right)$, where the value of z is the expected amount of time in hours that she waits if she arrives at time x. The average value of z is then $\frac{1}{4}$; translating z to minutes, we get $z = 15$. Computing y similarly, we find $y = 35$. Thus, the answer is $35 \cdot 15 = \boxed{525}$.

   Problem written by Bill Huang and Zhuo Qun Song.

8. We claim that Ledecky performs objectively better than Phelps if and only if $g_L \geq g_P$, $g_L + s_L \geq g_P + s_P$, and $g_L + s_L + b_L \geq g_P + s_P + b_P$, but it is not the case that $g_L = g_P$, $s_L = s_P$, and $b_L = b_P$. First, assume that all these conditions are satisfied. Let $0 < w_b < w_s < w_g$ be the weights and let $m_b = w_b$, $m_s = w_s - w_b$, and $m_g = w_g - w_s$ (all positive values). Note that

$$w_g g_L + w_s s_L + w_b b_L = m_b(b_L + s_L + g_L) + m_s(s_L + g_L) + m_g g_L$$

   and similarly for Phelps. Thus, $w_g g_L + w_s s_L + w_b b_L > w_g g_P + w_s s_P + w_b b_P$.

   Now assume that at least one of these conditions fails. If $g_L < g_P$ then letting $w_g = 1$, $w_s = .02$, and $w_b = .01$ makes Ledecky's score lower than Phelps's. If $g_L + s_L \geq g_P + s_P$ then letting $w_g = 1$, $w_2 = .99$, and $w_b = .01$ makes Ledecky's score lower than Phelps's. If $g_L + s_L + b_L \geq g_P + s_P + b_P$ then letting $w_g = 1$, $w_2 = .99$, and $w_b = .98$ makes Ledecky's score lower than Phelps's. Finally, if $g_L = g_P$, $s_L = s_P$, and $b_L = b_P$, then Ledecky's score equals Phelps's regardless of the weights.

   Now for the computation. Give Ledecky and Phelps each an extra silver and bronze medal; this doesn't change the calculus. Note that (g_L, s_L, b_L) can now be represented by placing three bars in different places among twelve stars representing the events. For example, the configuration $** \mid * \mid ***** \mid *$ represents that Ledecky got 2 gold, 1 silver, and 5 bronze medals, and did not receive a medal in one event. We can represent (g_L, s_L, b_L) and (g_P, s_P, b_P) in this manner and then smoosh the two representations together, removing duplicate bars (ones that are in the same place). The resulting figure has at least 3 and at most 6 bars among 9 stars.

To do our counting, be go in the other direction: for every smooshed representation we find the number of representations for Ledecky and Phelps whose smooshing gives the desired smooshed representation, where Ledecky objectively outperforms Phelps. We do this using casework based on the number of bars in the smooshed representation.

Case 1: 3 bars. Then Ledecky and Phelps got the same number of gold, silver, and bronze medals, so Ledecky could not have objectively outperformed Phelps.

Case 2: 4 bars. Then Ledecky and Phelps have two bars in common and each have one bar that the other doesn't. For each configuration, this gives 12 possibilities, and the ones in which Ledecky objectively outperforms Phelps are the ones in which her bar comes first. So there are 6 possibilities. There are $\binom{10}{4}$ ways to place four bars among nine stars so that none of the bars are next to each other.

Case 3: 5 bars. There are 5 ways to choose the bar in common. Among the other four bars, the order must be LLPP or LPLP, so there are 10 possibilities. There are $\binom{10}{5}$ ways to place five bars among nine stars so that none of the bars are next to each other.

Case 4: 6 bars. The order must be LLLPPP, LLPPLP, LPLLPP, LPLPLP, or LLPLPP, so there are 5 possibilities. There are $\binom{10}{6}$ ways to place six bars among nine stars so that none of the bars are next to each other.

Thus, the answer is $6\binom{10}{4} + 10\binom{10}{5} + 5\binom{10}{6} = \boxed{4830}$.

Problem written by Eric Neyman.

If you believe that any of these answers is incorrect, or that a problem had multiple reasonable interpretations or was incorrectly stated, you may appeal at `tinyurl.com/pumacappeals`. All appeals must be in by 1 PM to be considered.