1. Let $T$ be the point of tangency and $D$ be the intersection of $AT$ and $BC$; note that $AD$ is an altitude. Since $\angle BTC = 120^\circ$, we find that $TD = \frac{1}{2} \sqrt{3} = \frac{AD}{3}$. Let $K$ be the intersection of $AT$ and $PQ$; since $\angle APK = \angle PTK = 60^\circ$, we have $AK = \frac{3}{1+3} \cdot (1 - \frac{1}{3}) \cdot AD = \frac{4D}{2}$, so $P, Q$ are the midpoints of $AB, AC$, respectively. The area of $\triangle APQ$ is then $\sqrt{3} \frac{16}{16}$, so $a + b = 19$.

Problem written by Mel Shu and Bill Huang.

2. We will reference the following picture (which we need when we use words like “horizontal” and “vertical”).

Let $R$ be the radius of $\omega$. The distance from $M$ to the midpoint of $AD$ can be expressed as $R + \sqrt{R^2 - 4^2}$, and this is equal to 8. We thus have $R^2 - 4^2 = (8 - R)^2$, and solving for $R$ gives $R = 5$. The distance from $O$ to $AB$ is 4, so the Pythagorean theorem gives us that the vertical distance from $O$ to $X$ is 3. Thus, if we use a coordinate system with the natural $x$ and $y$ axes and origin $O$, then the line $OX$ is described by the equation $y = \frac{3}{4} x$. Meanwhile, the line $AM$ has slope 2 and $y$-intercept 5 so it is described by the line $y = 2x + 5$. The intersection point $Y$ thus has $x$-coordinate satisfying $2x + 5 = \frac{3}{4} x$, i.e. $x = \frac{20}{11}$, and this gives $y = \frac{15}{11}$. Thus, we have $OY = \sqrt{\left(\frac{-20}{11}\right)^2 + \left(\frac{15}{11}\right)^2} = \frac{25}{11}$.

Our answer is thus $25 + 11 = 36$.

Problem written by Eric Neyman.

3. The net of the curved surface of $C$ is a sector of a disc. Denote by $P'$ the point on the disc corresponding to the point $P$ on $C$. Then the shortest path along the curved surface of $C$ between $P_1$ and $P_3$ corresponds to the line segment $P'_1P'_2$. By symmetry, $P'_1P'_3 \perp A'P'_2$. Thus $P'_1$ lies on the perpendicular bisector of $A'P'_2$, so $P'_1P'_2 = P'_1A' = P'_2A'$ and $\triangle A'P'_1P'_2$ is equilateral ($A'$ is the center of the disc). Then if $s = AP_1 = A'P'_1$ is the slanted height of $C$, we get $2\pi r = \frac{k}{k} 2\pi s \Rightarrow s = \frac{9r}{5}$. Then $\left(\frac{k}{r}\right)^2 = \frac{k^2}{r^2} = \frac{s^2 - r^2}{r^2} = \frac{11}{25}$ so $a + b = 11 + 25 = 36$.

Problem written by Mel Shu.
4. The length of the altitude from $I$ to $BC$, i.e. the inradius $r$, equals the length of the altitude from $G$ to $BC$, which is one-third the height of the triangle, which we will call $h$. We thus have

$$[ABC] = \frac{ah}{2} = \frac{rp}{2} = \frac{hp}{6},$$

where $a = BC = 2016$ and $p$ is the perimeter of the triangle, so $p = 3a$, and indeed if $p = 3a$ then the conditions of the problem are satisfied. This means that $A$ must lie on the ellipse with foci $B$ and $C$ such that $AB + AC = 2a$, implying a major axis of length $2a$. Thus, $AB < \frac{3a}{2} = 3024$. Since $AB$ is an integer, $AB$ is at most $3023$.

*Problem written by Bill Huang.*

5. First of all, $BF \cdot BA = BD \cdot BC \Rightarrow DC = 4$. We observe that $AE \cdot AC = AF \cdot AB = 28 \cdot 63 = 1764 = AF^2$. Thus $\angle APC = 90^\circ$, and similarly, $CP^2 = CE \cdot CA = CD \cdot CB = 196$. Therefore $CP = 14$.

*Problem written by Mel Shu.*

6. Let $BP = 1, CP = y, AP = x$. By the law of cosines we have:

$$1^2 + x^2 + \sqrt{2} \cdot 1 \cdot x = BC^2 = (2 \cdot BP)^2 = 2^2$$

$$1^2 + y^2 + \sqrt{2} \cdot 1 \cdot y = AB^2 = AC^2 = x^2 + y^2$$

Solving (using the fact $x > 0$) yields $x = \frac{\sqrt{7} - 1}{\sqrt{2}}$ and $y = \frac{3 - \sqrt{7}}{\sqrt{2}}$. Then, $\frac{CP}{AP} = \frac{x}{y} = 2 + \sqrt{7}$, so $a + b = 9$.

*Problem written by Bill Huang.*

7. Let $T$ be the foot of the altitude from $A$ to $BX$. Observe that $AT = 8$ and $BT = 6$ (integer Pythagorean triples). Then, $[DEY] = \frac{AT \cdot Ey}{2}$; it remains to find $Ey = AE - AY$.

Observe that $\triangle DCX \sim \triangle ABX$, and let $r$ be the ratio of similarity; we can write $CD = 10r$, $DX = 17r$ and $CX = 21r$. Then:

$$AE = XD \cdot \frac{CA}{CX} = 17r \cdot \frac{21r + 17}{21r}$$

Since $ABDY$ is an isosceles trapezoid, $AY = BD - 2BT = (21 + 17r) - 12 = 17r + 9$. Thus:

$$Ey = AE - AY = \frac{17}{21}(21r + 17) - (17r + 9) = \frac{17^2}{21} - 9 = \frac{100}{21}$$

Therefore, $[DEY] = \frac{1}{2} \cdot 8 \cdot \frac{100}{21} = \frac{400}{21}$, so $m + n = 421$.

*Problem written by Eric Neyman.*

8. Let $H$ be the orthocenter of $ABC$; since the reflection $H'$ of $H$ over $BC$ lies on $(ABC)$, the reflection $H''$ of $H$ over $M$ lies on $(ABC)$. Since $AH''$ is a diameter of $(ABC)$, it follows that $M,H,P$ are collinear. We then have $ADEHP$ concyclic. Let $F$ be the foot of the altitude from $A$ and $M'$ the reflection of $M$ over $F$. Angle chasing, $\angle CDX, BEY = \angle AEP, ADP = \angle AHP = \angle FHM = \angle M'HF$. Then, $\angle XDH, YEH = 90 - \angle M'HF = 180 - \angle HM'X, HM'Y$, so $DHM'X, EHM'Y$ are concyclic. By power of a point:

$$BM' \cdot BX = BH \cdot BD = BE \cdot BA \quad CM' \cdot CY = CH \cdot CE = CD \cdot CA$$

By the Pythagorean theorem, we find that:

$$BE = \frac{27}{8}, BM' = \frac{3}{2} \quad CD = \frac{9}{2}, CM' = \frac{9}{2}$$
This gives us $BX = 9, CY = 5$, and $XY = 9 + 5 - 6 = 8$. A quick calculation gives $AF = \frac{5\sqrt{7}}{4}$, so the area of $\triangle AXY$ is $5\sqrt{7}$. Thus, the square of the area is $175$.

*Problem written by Bill Huang.*

If you believe that any of these answers is incorrect, or that a problem had multiple reasonable interpretations or was incorrectly stated, you may appeal at tinyurl.com/pumacappeals. All appeals must be in by 1 PM to be considered.