PUMaC

Number Theory A

1. What is the smallest positive integer n such that $2016n$ is a perfect cube?

2. For positive integers i and j, define $d_{(i,j)}$ as follows: $d_{(1,j)} = 1$, $d_{(i,1)} = 1$ for all i and j, and for $i, j > 1$, $d_{(i,j)} = d_{(i-1,j)} + d_{(i,j-1)} + d_{(i-1,j-1)}$. Compute the remainder when $d_{(3,2016)}$ is divided by 1000.

3. For odd positive integers n, define $f(n)$ to be the smallest odd integer greater than n that is not relatively prime to n. Compute the smallest n such that $f(f(n))$ is not divisible by 3.

4. Compute the sum of the two smallest positive integers b with the following property: there are at least ten integers $0 \leq n < b$ such that n^2 and n end in the same digit in base b.

5. Let $k = 2^6 \cdot 3^5 \cdot 5^2 \cdot 7^3 \cdot 53$. Let S be the sum of $\frac{\gcd(m,n)}{\operatorname{lcm}(m,n)}$ over all ordered pairs of positive integers (m, n) where $mn = k$. If S can be written in simplest form as $\frac{r}{s}$, compute $r + s$.

6. Find the sum of the four smallest prime divisors of $2016^{239} - 1$.

7. Compute the number of positive integers n between 2017 and 2017^2 such that $n^n \equiv 1 \pmod{2017}$. (2017 is prime.)

8. Let $n = 2^8 \cdot 3^9 \cdot 5^{10} \cdot 7^{11}$. For k a positive integer, let $f(k)$ be the number of integers $0 \leq x < n$ such that $x^2 \equiv k^2 \pmod{n}$. Compute the number of positive integers k such that $k \mid f(k)$.