Team Round

Do not turn this page over until you are instructed to do so by your proctor. Before you begin the test, you and your team have 5 minutes to read and discuss this page. After those 5 minutes are over, you will have 25 minutes to complete the team round itself.

For each problem, you will be asked to wager an integer number of tokens between 0 and 100, inclusive. For the m-th token you wager, you get $100 - m$ bonus points if you get the correct answer to the problem and you lose m points if you get the incorrect answer. Then, your bonus points are multiplied by $\frac{w}{10000}$, where w is the number of points that the problem is worth (which will be indicated next to each problem).

For example, suppose that you wager 5 coins on a problem worth 8 points.

If you get the problem right, you get

$$\frac{8}{10000}(99 + 98 + 97 + 96 + 95) = 0.388$$

bonus points. On the other hand, if you get the problem wrong, you lose

$$\frac{8}{10000}(1 + 2 + 3 + 4 + 5) = 0.012$$

bonus points.

To compute your score on the team round, all of your bonus points are added up (summing over the 15 problems), and your bonus score on the team round will be this number, but no less than 0. (So if your total bonus points are negative, you will receive no bonus score.) Your final team round score will be the number of points you get from solving the problems, plus your bonus score.

If for any problem you do not write down the number of tokens you wish to wager on your answer sheet, it will be assumed that you wish to wager 0 tokens.

Good luck!

1. **(3)** Quadrilateral $ABCD$ has integer side lengths, and angles ABC, ACD, and BAD are right angles. Compute the smallest possible value of AD.

2. **(3)** Temerant is a spherical planet with radius 1000 kilometers. The government wants to build twelve towers of the same height on the equator of Temerant, so that every point on the equator can be seen from at least one tower. The minimum possible height of the towers can be written, in kilometers, as $a\sqrt{b} - c\sqrt{d} - e$ for positive integers a, b, c, d, and e (with b and d not divisible by the square of any prime). Compute $a + b + c + d + e$.

3. **(4)** Compute the sum of all positive integers $n < 200$ such that $\gcd(n, k) \neq 1$ for every $k \in \{2 \cdot 11 \cdot 19, 3 \cdot 13 \cdot 17, 5 \cdot 11 \cdot 13, 7 \cdot 17 \cdot 19\}$.

4. **(4)** For $x > 1$, let $f(x) = \log_2(x + \log_2(x + \log_2(x + \dots)))$. Compute

$$\sum_{k=2}^{10} f^{-1}(k).$$

5. **(4)** An alphabet A has 16 letters. A message is written using the alphabet and, to encrypt the message, a permutation $f : A \to A$ is applied to each letter. Let $n(f)$ be the smallest positive integer k such that every message m, encrypted by applying f to the message k times, produces m. Compute the largest possible value of $n(f)$.

6. **(5)** Compute the sum of all positive integers less than 100 that do not have consecutive 1s in their binary representation.

7. **(5)** In triangle ABC, let S be on BC and T be on AC so that $AS \perp BC$ and $BT \perp AC$, and let AS and BT intersect at H. Let O be the center of the circumcircle of $\triangle AHT$, P be the center of the circumcircle of $\triangle BHS$, and G be the other point of intersection (besides H) of the two circles. Let GH and OP intersect at X. If $AB = 14$, $BH = 6$, and $HA = 11$, then $XO - XP$ can be written in simplest form as $\frac{m}{n}$. Find $m + n$.

8. **(6)** Alice has 100 balls and 10 buckets. She takes each ball and puts it in a bucket that she chooses at random. After she is done, let b_i be the number of balls in the ith bucket, for $1 \leq i \leq 10$. Compute the expected value of $\sum_{i=1}^{10} b_i^2$.

9. **(6)** Let $\triangle ABC$ be a right triangle with $AB = 4$, $BC = 5$, and hypotenuse AC. Let I be the incenter of $\triangle ABC$ and E be the excenter of $\triangle ABC$ opposite A (the center of the circle tangent to BC and the extensions of segments AB and AC). Suppose the circle with diameter IE intersects line AB beyond B at D. If $BD = \sqrt{a} - b$, where a and b are positive integers. Find $a + b$.

10. **(8)** Chad and Chad2 run competing rare candy stores at Princeton. Chad has a large supply of boxes of candy, each box containing three candies and costing him \$3 to purchase from his supplier. He charges \$1.50 per candy per student. However, any rare candy in an opened box must be discarded at the end of the day at no profit. Chad knows that at each of 8am, 10am, noon, 2pm, 4pm, and 6pm, there will be one person who wants to buy one candy, and that they choose between Chad and Chad2 at random. (He knows that those are the only times when he might have a customer.) Chad may refuse sales to any student who asks for candy. If Chad acts optimally, his expected daily profit can be written in simplest form as $\frac{m}{n}$. Find $m + n$. (Chad's profit is \$1.50 times the number of candies he sells, minus \$3 per box he opens.)

11. **(8)** Madoka chooses 4 random numbers a, b, c, d between 0 and 1. She notices that $a+b+c=1$. If the probability that $d > a, b, c$ can be written in simplest form as $\frac{m}{n}$, find $m+n$.

12. **(10)** King Tin writes the first n perfect squares on the royal chalkboard, but he omits the first (so for $n = 3$, he writes 4 and 9). His son, Prince Tin, comes along and repeats the following process until only one number remains:

    > He erases the two greatest numbers still on the board, calls them a and b, and writes the value of $\frac{ab-1}{a+b-2}$ on the board.

    Let $S(n)$ be the last number that Prince Tin writes on the board. Let

    $$\lim_{n\to\infty} S(n) = r,$$

    meaning that r is the unique number such that for every $\epsilon > 0$ there exists a positive integer N so that $|S(n) - r| < \epsilon$ for all $n > N$. If r can be written in simplest form as $\frac{m}{n}$, find $m+n$.

13. **(10)** Ayase randomly picks a number $x \in (0, 1]$ with uniform probability. He then draws the six points $(0,0,0), (x,0,0), (2x,3x,0), (5,5,2), (7,3,0), (9,1,4)$. If the expected value of the volume of the convex polyhedron formed by these six points can be written as $\frac{m}{n}$ for relatively prime positive integers m and n, find $m+n$.

14. **(12)** Suppose $P(x) = x^{2016} + a_{2015}x^{2015} + \ldots + a_1 x + a_0$ satisfies

    $$P(x)P(2x+1) = P(-x)P(-2x-1)$$

    for all $x \in \mathbb{R}$. Find the sum of all possible values of a_{2015}.

15. **(12)** Compute the sum of all positive integers n with the property that $x^n \equiv 1 \pmod{2016}$ has n solutions in $\{0, 1, 2, \ldots, 2015\}$.