P U M ∴ C

Team Round Solutions

1. Let $AB = a$, $BC = b$, and $AC = c$. Then by similar triangles, $CD = c \cdot \frac{a}{b}$ and $AD = \frac{c^2}{b}$. (a, b, c) must be a Pythagorean triple, from which it can easily be found that the smallest possible value of $\frac{c^2}{b}$ is $\boxed{25}$.

   Problem written by Eric Neyman.

2. Each tower needs to see an arc of measure $30°$, i.e. an $15°$ arc on either side, so that every point is covered twice. This means that $\frac{1000}{1000+h} = \cos 15°$, where h is the height of the towers. Note that $\cos 15° = \cos(45° - 30°) = \frac{\sqrt{2}}{2}\left(\frac{\sqrt{3}}{2} + \frac{1}{2}\right) = \frac{\sqrt{6}+\sqrt{2}}{4}$, and so we have $1 + \frac{h}{1000} = \frac{1000+h}{1000} = \frac{4}{\sqrt{6}+\sqrt{2}} = \sqrt{6} - \sqrt{2}$. Thus, $h = 1000\sqrt{6} - 1000\sqrt{2} - 1000$, and so our answer is $1000 + 6 + 1000 + 2 + 1000 = \boxed{3008}$.

   Problem written by Eric Neyman.

3. Clearly n must have factors in each of $\{2, 11, 19\}$, $\{3, 13, 17\}$, $\{5, 11, 13\}$, and $\{7, 17, 19\}$. If $19 \mid n$ then $\frac{n}{19}$ must have a factor in common with both $3 \cdot 13 \cdot 17$ and $5 \cdot 11 \cdot 13$, so n is at least $19 \cdot 13$, which is impossible since $n < 200$. If $17 \mid n$ then $\frac{n}{17}$ must have a factor in common with both $2 \cdot 11 \cdot 19$ and $5 \cdot 11 \cdot 13$, and note that $\frac{n}{17} \leq 11$. This gives two possibilities: $17 \cdot 10 = 170$ and $17 \cdot 11 = 187$. If n is divisible by 13 then $\frac{n}{13}$ must have a factor in common with both $2 \cdot 11 \cdot 19$ and $7 \cdot 17 \cdot 19$. This gives $n = 13 \cdot 14 = 182$. If n does not divide any of 13, 17, and 19 then it is easy to see that n must be divisible by 3 and 7. Then n must also have a factor in common with $2 \cdot 11 \cdot 19$ and $5 \cdot 11 \cdot 13$, which means that $\frac{n}{21} \geq 10$, which is impossible. Thus, the answer is $170 + 182 + 187 = \boxed{539}$.

   Problem written by Eric Neyman.

4. If $f(x) = k$, we have $2^k - x = k$, so $x = 2^k - k$. Thus, we have

   $$\sum_{k=2}^{10} f^{-1}(k) = \sum_{k=2}^{10}(2^k - k) = \sum_{k=2}^{10} 2^k - \sum_{k=2}^{10} k = 2^{11} - 4 - 54 = 2048 - 58 = \boxed{1990},$$

   as desired.

   Problem written by Eric Neyman.

5. Write f in cycle form. Then $n(f)$ is the least common multiple of the cycle lengths. The cycle lengths add up to 16. Observe that powers of primes are the most efficient, because any non-power-of-primes can be split into its prime-power components. We do casework by largest cycle length:

   - $13 \implies 13 \cdot 3 = 39$ is the largest possible $n(f)$.
   - $11 \implies 11 \cdot 3 \cdot 2 = 66$.
   - $9 \implies 9 \cdot 2 \cdot 5 = 90$.
   - $8 \implies 8 \cdot 3 \cdot 5 = 120$.
   - $7 \implies 7 \cdot 5 \cdot 4 = 140$.

   So the maximum is $\boxed{140}$.

   Problem written by Eric Neyman.

6. This is the same as the sum of all 7-digit binary sequences with no consecutive 1s. We note that the number of d-digit binary sequences with this property goes 2, 3, 5, 8, 13 for $d = 1, 2, 3, 4, 5$; let $n(d)$ represent this sequence. Let $f(d)$ be the sum-sequence. Then $f(1) = 1$, $f(2) = 3$, and for $d > 2$, we have $f(d) = 4f(d-2) + n(d-2) + n(d-1)$ (the first two terms are if the sequence we're adding in ends in a 1 and the last term is if it ends in a 0). Evaluating this for $d = 3, 4, 5, 6$, and, finally, 7, gives $f(7) = \boxed{1389}$, as desired.

   Problem written by Eric Neyman.

7. Since AHT and BHS are right triangles, O is the midpoint of AH and P is the midpoint of BH. This means that $\angle BGH$ and $\angle AGH$ are right angles, so G is the foot of the altitude from C to AB. Note also that $\triangle HOP \sim \triangle HAB$ with factor $\frac{1}{2}$. We thus have

$$\begin{aligned} XO - XP &= \frac{XO^2 - XP^2}{XO - XP} = \frac{XO^2 - XP^2}{7} = \frac{(HO^2 - HX^2) - (HP^2 - HX^2)}{7} \\ &= \frac{HO^2 - HP^2}{7} = \frac{11^2 - 6^2}{28} = \frac{85}{28}, \end{aligned}$$

   so our answer is $85 + 28 = \boxed{113}$.

   Problem written by Eric Neyman.

8. Observe that for each ball, the expected number of balls in its bucket is $1 + 99 \cdot \frac{1}{10} = \frac{109}{10}$. The expected value of $\sum_{i=1}^{1} 0b_i^2$ is just the expected value of the sum over all balls of the number of balls in their bucket. By linearity of expectation, this is equal to 100 times the expected number of balls in a ball's bucket. Thus, the answer is $100 \cdot \frac{109}{10} = \boxed{1090}$.

   Problem written by Eric Neyman.

9. Note that $AC = \sqrt{41}$. An angle chase shows that $BICED$ is cyclic, and the reflection across AI takes D to C. Therefore $AD = AC = \sqrt{41}$, so $BD = \sqrt{41} - 4$. Thus, the answer is $41 + 4 = \boxed{45}$.

10. If there's 2 students left, Chad shouldn't open another box, because the chance that he can make \$1.5 from the box is $\frac{1}{4}$, the chance he makes \$0 is $\frac{2}{4}$, the chance he makes -\$1.5 is $\frac{1}{4}$, so there's no expected gain. If there are any more students left, then he should continue to offer sales. We can thus separate consideration of the first 3 students (for which Chad always opens a box) and the last 3 students (for which Chad never opens a box).

    Of the first 3, if 0 go to Chad, his profit is 0.

    If 1, his profit is 1.5 if at least two of the last three go to him (with probability $\frac{1}{2}$), 0 if exactly one go to him, and -1.5 if none do (with probability $\frac{1}{8}$). The total expected profit is $\frac{1}{2} \cdot 1.5 - \frac{1}{8} \cdot 1.5 = \frac{9}{16}$.

    If 2, his profit is 1.5 if at least one of the last three go to him, and 0 otherwise. The to tal expected profit is $\frac{7}{8} \cdot 1.5 = \frac{21}{16}$.

    If 3, his profit is 1.5.

    So his expected profit is $\frac{3}{8} \cdot \frac{9}{16} + \frac{3}{8} \cdot \frac{21}{16} + \frac{1}{8} \cdot \frac{3}{2} = \frac{57}{64}$. Thus, the answer is $57 + 64 = \boxed{121}$.

11. The answer is $1 - \mathbb{E}(\max(a, b, c))$, where $\mathbb{E}$ denotes the expected value. We deal with only the case where $a > b, c$, and split it into two further cases.

    Case 1: $a < \frac{1}{2}$. The probability this occurs is $\frac{1}{12}$. The expected value of a is $\frac{4}{9}$.

    Case 2: $a > \frac{1}{2}$. The probability this occurs is $\frac{1}{4}$. The expected value of a is $\frac{2}{3}$.

    In total, $\mathbb{E}(\max(a, b, c)) = 3\left(\frac{1}{12}\frac{4}{9} + \frac{1}{4}\frac{2}{3}\right) = \frac{11}{18}$. The probability is thus $\frac{7}{18}$, so the answer is $7 + 18 = \boxed{25}$.

12. Define the operation $\wr$ by $a \wr b = \frac{ab-1}{a+b-2}$. Then, noticing the similarity between $\frac{ab-1}{a+b-2}$ and $\frac{ab}{a+b} = \frac{1}{\frac{1}{a}+\frac{1}{b}}$, we find that

$$a \wr b = 1 + \frac{1}{\frac{1}{a-1} + \frac{1}{b-1}},$$

which is both commutative and associative. Therefore,

$$S(n) = 4 \wr 9 \wr \cdots \wr n^2 = 1 + \left(\sum_{k=2}^{n} \frac{1}{k^2-1} \right)^{-1},$$

so

$$\lim_{n\to\infty} S(n) = 1 + \left(\sum_{k=2}^{\infty} \frac{1}{k^2-1} \right)^{-1} = 1 + \left(\sum_{k=2}^{\infty} \frac{1/2}{k-1} - \frac{1/2}{k+1} \right)^{-1} = \frac{7}{3},$$

and our answer is $7 + 3 = \boxed{10}$.

Problem written by Matt Tyler.

13. Note that each of these six points lie on a different edge of the tetrahedron with vertices $(0,0,0), (10,0,0), (4,6,0), (8,2,8)$, which has area $A = \frac{1}{6} \cdot 10 \cdot 6 \cdot 8 = 80$. The resulting polygon is essentially this tetrahedron with the vertices cut off, and finding the volume of the cut-off pieces is simple. Label the four vertices of the tetrahedron $A_1, \dots, A_4$ and the six points Ayase draws $B_1, \dots, B_6$, respectively. Then:

$$\begin{aligned}
\mathbb{E}[A_1B_2B_3B_1] &= [A_1A_2A_3A_4] \cdot \mathbb{E}\left(\frac{A_1B_2}{A_1A_2} \cdot \frac{A_1B_3}{A_1A_3} \cdot \frac{A_1B_1}{A_1A_4} \right) \\
&= 80 \cdot \mathbb{E}\left(\frac{x}{10} \cdot \frac{x}{2} \cdot \frac{0}{2\sqrt{33}} \right) = 0 \\
\mathbb{E}[A_2B_2B_5B_6] &= [A_2A_1A_3A_4] \cdot \mathbb{E}\left(\frac{A_2B_2}{A_2A_1} \cdot \frac{A_2B_5}{A_2A_3} \cdot \frac{A_2B_6}{A_2A_4} \right) \\
&= 80 \cdot \mathbb{E}\left(\frac{10-x}{10} \cdot \frac{1}{2} \cdot \frac{1}{2} \right) = 19 \\
\mathbb{E}[A_3B_3B_5B_4] &= [A_3A_1A_2A_4] \cdot \mathbb{E}\left(\frac{A_3B_3}{A_3A_1} \cdot \frac{A_3B_5}{A_3A_2} \cdot \frac{A_3B_4}{A_3A_4} \right) \\
&= 80 \cdot \mathbb{E}\left(\frac{\sqrt{13}(2-x)}{2\sqrt{13}} \cdot \frac{1}{2} \cdot \frac{1}{4} \right) = \frac{15}{2} \\
\mathbb{E}[A_4B_1B_6B_4] &= [A_4A_1A_2A_3] \cdot \mathbb{E}\left(\frac{A_4B_1}{A_4A_1} \cdot \frac{A_4B_6}{A_4A_2} \cdot \frac{A_4B_4}{A_4A_3} \right) \\
&= 80 \cdot \mathbb{E}\left(\frac{1}{1} \cdot \frac{1}{2} \cdot \frac{3}{4} \right) = 30
\end{aligned}$$

Which yields:

$$\mathbb{E}[B_1B_2B_3B_4B_5B_6] = 80 - 0 - 19 - \frac{15}{2} - 30 = \frac{47}{2}.$$

Thus, the answer is $47 + 2 = \boxed{49}$.

Problem written by Bill Huang.

14. Observe that the condition implies that the multisets of the roots of the two polynomials $P(x)P(2x+1)$ and $P(-x)P(-2x-1)$ are identical.

Let $R = (r \in \mathbb{R}, P(r) = 0)$ be the multiset of roots of P. Given $r \in R$, if $r' = -r \in R$, then we can cancel out the instances of r in the LHS with the instances of r' in the RHS. We can then let $R' = R/(r, r')$. We repeat this removal of pairs (r, r') until no such pair remains, and call the resulting set S.

Now, for any $r \in S$, r must be a root of $P(-2x-1)$, so there exists $s \in S$ such that $r = -2s - 1 \leftrightarrow s = \frac{-r-1}{2}$. Taking an arbitrary $r_0 \in S$, we inductively define $r_{n+1} = \frac{-r_n-1}{2}$ for $n \geq 0$, and indeed, $r_{n+1} \in S$. This gives us for arbitrary $k \geq 0$:

$$r_k = \frac{(-1)^k r_1 - \frac{2^k-(-1)^k}{3}}{2^k}$$

Since S is finite, there exists some n for which $r_n = r_0$, and solving yields $r_0 = -\frac{1}{3}$. Hence, S contains only roots equal to $-\frac{1}{3}$. Then:

$$a_{2015} = -\sum_{r \in R} r = -\sum_{r \in S} r = \frac{|S|}{3}$$

Since $|S| \in \{2016, 2014, \dots 0\}$, the sum of all possible values of a_{2015} is:

$$\sum = \frac{1}{3} \cdot (2016 + 2014 + \dots + 0) = \frac{1008 \cdot 1009}{3} = \boxed{339024}.$$

Problem written by Bill Huang.

15. Note that $x^n \equiv 1 \pmod{2016}$ if and only if $x^n \equiv 1 \pmod{32}$ and $x^n \equiv 1 \pmod 9$ and $x^n \equiv 1 \pmod 7$. Write the integers modulo 7 relatively prime to 7 as $\{1, 3, 3^2, 3^3, 3^4, 3^5\}$. If $x \equiv 3^k \pmod 7$ then $x^n \equiv 1 \pmod 7$ if and only if $kn \equiv 1 \pmod 6$, which happens for $\gcd(n, 6)$ values of k (and therefore x) modulo 7. The same holds modulo 9, because we can write the integers modulo 9 relatively prime to 9 as $\{1, 2, 2^2, 2^3, 2^4, 2^5\}$ (so again there are $\gcd(n, 6)$ possible values of x modulo 9).

    The value of x modulo 32 is more difficult to deal with. We find that the numbers modulo 32 relatively 32 can be written as $\{3^k, 5 \cdot 3^k \mid k \in \{0, 1, \dots, 7\}\}$. From this structure we find that there is one element of order 1, three of order 2, four of order 4, and eight of order 8. Thus, if n is odd there is 1 value of x modulo 32; 4 if n is divisible by 2 but not 4; 8 if n is divisible by 4 but not 8; and 16 if n is divisible by 8.

    Thus, the total number of values of x in $\{0, 1, \dots, 2015\}$ is $\gcd^2(n, 6)$ times either 1, 4, 8, or 16; and this value must equal n, so n is of the form $2^k 3^m$. From our formula for the number of values of x we conclude that m is either 0 or 2 and k is either 0 or 6. Thus, the answer is $1 + 9 + 64 + 64 \cdot 9 = (1 + 9)(1 + 64) = \boxed{650}$.

    Problem written by Eric Neyman.

If you believe that any of these answers is incorrect, or that a problem had multiple reasonable interpretations or was incorrectly stated, you may appeal at `tinyurl.com/pumacappeals`. All appeals must be in by 1 PM to be considered.