P U M ∴ C

Individual Finals A Solutions

1. Arrange the 12 candies in a circle. We pair up the four rare candies with non-rare candies in the following manner: choose a rare candy that doesn't have a rare candy immediately to its left, pair it up with the candy immediately to its left, and remove them from the circle. Note that the ordering of the choice of rare candies doesn't matter: each rare candy is identified with the candy originally $2n-1$ candies to the left such that the $2n-2$ candies in between them have half rare, and n is minimal. Chad's friend chooses the non-rare candy in each pair. Chad can identify the four rare candies by identifying the four pairings from the chosen candies: he identifies each chosen candy with the candy $2n-1$ candies to the right such that the $2n-2$ candies in between have half chosen and half unchosen, with n minimal.

   Problem written by Zhuo Qun Song.

2. From the defining equation we obtain $\lambda = k + \frac{c}{\lambda^m} \Rightarrow \lfloor \lambda n \rfloor = kn + \left\lfloor \frac{cn}{\lambda^m} \right\rfloor$ for all $n \in \mathbb{Z}^+$, so $f(n) \equiv \left\lfloor \frac{cn}{\lambda^m} \right\rfloor \pmod{k}$. Thus $f^{m+1}(n) \equiv \left\lfloor \frac{cf^m(n)}{\lambda^m} \right\rfloor \pmod{k}$ so it suffices to show that $(cn-1)\lambda^m \le cf^m(n) < cn\lambda^m$. We note that if $\lambda = \frac{a}{b}$ is rational with $\gcd(a,b) = 1$ then $b^m\lambda^{m+1}$ is an integer, so $b = \pm 1$ and λ is an integer. Therefore λ must be irrational, so $f(n) < \lambda n \Rightarrow f^m(n) < \lambda^m n \Rightarrow cf^m(n) < cn\lambda^m$. On the other hand, $f(n) > \lambda n - 1$ so $f^m(n) > \lambda(\lambda(\ldots\lambda(\lambda n - 1) - 1\ldots) - 1) - 1 = \lambda^m n - \frac{\lambda^m - 1}{\lambda - 1} \Rightarrow cf^m(n) > cn\lambda^m - c\frac{\lambda^m-1}{\lambda-1}$ where $c\frac{\lambda^m-1}{\lambda-1} \le (k-1)\frac{\lambda^m-1}{\lambda-1} < \lambda^m$. The second inequality holds because $(k-1)(\lambda^m - 1) < \lambda^m(\lambda - 1) \Leftrightarrow k\lambda^m - k + 1 < \lambda^{m+1} = k\lambda^m + c$ which is true since $k + c > 1$. This completes the proof.

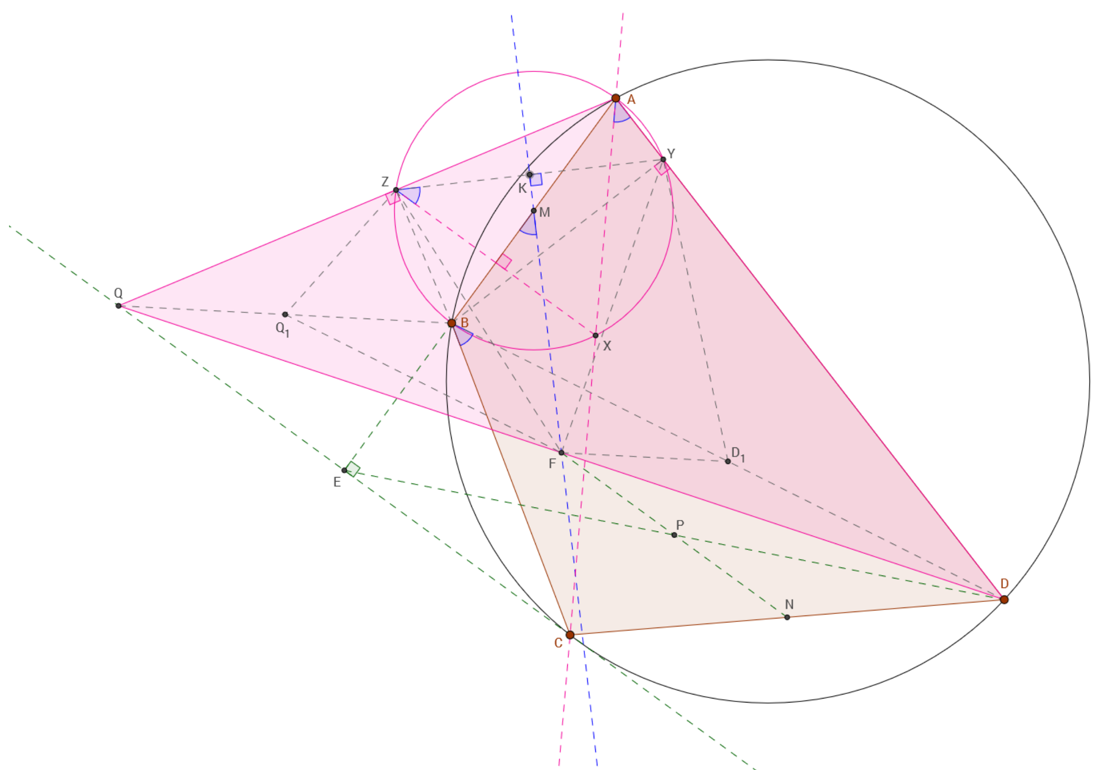
   Problem written by Mel Shu.

   (See next page for solution to Problem 3.)

3. Denote by ϕ reflection over AB. Now, let $Q = \phi(C)$; since $\triangle PDN \sim \triangle EDC$ with ratio $\frac{1}{2}$ and P, E are the midpoints of CQ, NF, $\triangle FDN \sim \triangle QDC$ with ratio $\frac{1}{2}$ so F is the midpoint of QD. Now, let the circle ω with diameter AB intersect AC at X and AD at Y, and let $Z = \phi(X)$. Since $Z \in \omega$, $\angle XZY = \angle XAY = \angle CAD = \angle CBD$. Furthermore, $XZ \perp AB$, so it suffices to show $YZ \perp MF$.

   Note that $BY \perp DY, BZ \perp QZ$. Now, let D_1, Q_1 be the midpoints of DB, QB, respectively. Since $\triangle DYB, \triangle QZB$ are right triangles and FQ_1D_1 is the medial triangle of BDQ, $YD_1 = BD_1 = Q_1F$ and $ZQ_1 = BQ_1 = D_1F$. Furthermore, $\angle BD_1F = \angle BQ_1F$ and $\angle BDA = \angle BCA = \angle BQA \longrightarrow \angle BD_1Y = \angle BQ_1Z$, so $\angle FD_1Y = \angle FQ_1Z$. Then, by SAS congruency, $\triangle FD_1Y \cong \triangle FQ_1Z$, which implies that $FY = FZ$.

   Let K be the midpoint of YZ. Since $\triangle FYZ$ is isosceles with base YZ, $FK \perp YZ$. Furthermore, since YZ is a chord of ω with center M, $MK \perp YZ$. Then, F, M, K are collinear and $MF \perp YZ$, so $\angle BMF = \angle XZY = \angle CBD$, as desired.



Problem written by Bill Huang.