Algebra B Solutions

1. We have $(x^2-1)^2 = 9801$, so $x^2 - 1 = \pm 99$. But x^2 cannot be negative, so $x^2 = 99+1 = 100$. Since x is positive, we have $x = \boxed{10}$.

   Problem written by Eric Neyman.

2. We have $a_2^3 + a_3^3 + a_4^3 = 4$, $a_5^3 + a_6^3 + a_7^3 = 7$, $a_8^3 + a_9^3 + a_{10}^3 = 10$, and $a_1^3 = 8000$. Thus, the answer is $\boxed{8021}$.

   Problem written by Eric Neyman.

3. The translation of the axes is equivalent to translating the polynomial 20 to the right and 16 down. Thus, the new polynomial has the equation

   $$y = (x-20)^3 - 13(x-20)^2 + 40(x-20) + 25 - 16.$$

   The sum of the roots of this polynomial is negative the coefficient of x^2, which is $-3 \cdot 20 - 13 = -73$. Thus, the answer is $\boxed{73}$.

   Alternatively, note that changing the constant coefficient does not change the sum of the roots, and then translating the polynomial 20 to the right increases the real part of each root by 20. Thus, the sum of the roots goes from 13 to $13 + 60 = 73$.

   Problem written by Eric Neyman.

4. Note that $f(f(f(f(f(x))))) = f(x)$ is a linear equation in x and thus has one solution. If $f(x) = x$ then clearly this equation is satisfied. Thus the solution is the solution to the equation $x = 15x - 2016$, which is $\boxed{144}$.

   Problem written by Eric Neyman.

5. We have

   $$\begin{aligned} 4096 f(f(x,x),x) &= x^{13} \\ 4096\left(x^{\log_2 x}\right)^{\log_2 x} &= x^{13} \\ 4096 x^{(\log_2 x)^2} &= x^{13} \\ 4096 x^{(\log_2 x)^2 - 13} &= 1 \\ 2^{(\log_2 x)((\log_2 x)^2 - 13) + 12} &= 2^0 \\ (\log_2 x)((\log_2 x)^2 - 13) + 12 &= 0 \\ (\log_2 x)^3 - 13\log_2 x + 12 &= 0 \\ (\log_2 x - 1)(\log_2 x - 3)(\log_2 x + 4) &= 0, \end{aligned}$$

   so x can be any of 2, 8, and $\frac{1}{16}$. Thus, the sum of all possible values of x is $\frac{161}{16}$, and so our answer is $161 + 16 = \boxed{177}$.

   Problem written by Eric Neyman.

6. Using the well known result that $x_1 - x_2 \mid P(x_1) - P(x_2)$ we get that $N \equiv 2 \pmod{2015}$, $N \equiv 3 \pmod{2014}$ and $N \equiv 3 \pmod{2013}$. Solving these equations gives $N \equiv 3 + 1007 \times 2013 \times 2014 \pmod{2013 \times 2014 \times 2015}$. Thus $N = 3 + 1007 \times 2013 \times 2014$ and $N \equiv 3 + 1007 \times (-3) \times (-2) \equiv 2013 \pmod{2016}$. The polynomial $P(x) = (x-1)((x-2)(1006)+1)+2$ satisfies all given conditions, so $\boxed{2013}$ is the final answer.

   Problem written by Mel Shu.

7. By a computation, $a_5 = 2^{12}-1$. If $a_i = 2^k-1$, then $a_{i+1} = (2^{k-1}-1)2^{k+1}$, so $a_{i+k+2} = 2^{k-1}-1$. Eventually we get $a_{105} = 0$. Thus the answer is $\boxed{105}$.

   Problem written by Zhuo Qun Song.

8. We first determine all solutions to the equation $P(x^2) = P(x)P(x-1)$. Let $P(x) = \prod(x-\alpha_i)$ where α_i are the roots of P (including multiplicity). Then $\prod(x^2-\alpha_i) = \prod(x-\alpha_i)\prod(x-(\alpha_i+1))$. Thus, considering the sets of roots of both sides of this equation, we get $\{\pm\sqrt{\alpha_i}\} = \{\alpha_i\} \cup \{\alpha_i+1\}$. Now we note that if $\max\left|\sqrt{\alpha_i}\right| = M > 1$ then $\max|\alpha_i| = M^2 > M$, contradiction. Similarly, if $\min\left|\sqrt{\alpha_i}\right| = N < 1$ then $\min|\alpha_i| = N^2 < N$, contradiction. (Unless $N = 0$. But this is impossible, since the multiplicity of zero as a root on both sides of the equation cannot be equal.) Therefore all roots α_i must satisfy $|\alpha_i| = |\alpha_i+1| = 1$, so $\alpha_i = \pm\left(\frac{-1+\sqrt{3}i}{2}\right)$. Now counting the distinct roots, the two different roots must have the same multiplicity, so the solutions are $P(x) = (x^2+x+1)^n$ for some positive integer n, or $P(x) = 0$ or $P(x) = 1$ (constant solutions). It follows that $P_0(1) = 3^n$, so $P_0(x) = (x^2+x+1)^2$. Therefore $P_0(10) = 111^2 = \boxed{12321}$.

   Problem written by Mel Shu.

If you believe that any of these answers is incorrect, or that a problem had multiple reasonable interpretations or was incorrectly stated, you may appeal at `tinyurl.com/pumacappeals`. All appeals must be in by 1 PM to be considered.