P U M ∴ C

Combinatorics B

1. Two fair six-sided dice are rolled. The probability that the positive difference between the two rolls is at least 4 can be written in simplest form as $\frac{m}{n}$. Compute $m+n$.

2. Every day, Kaori flips a fair coin. She practices her violin if and only if the coin comes up heads. The probability that she practices at least five days this week can be written in simplest form as $\frac{m}{n}$. Compute $m+n$.

3. Chitoge is painting a cube; she can paint each face either black or white, but she wants no vertex of the cube to be touching three faces of the same color. In how many ways can Chitoge paint the cube? Two paintings of a cube are considered to be the same if you can rotate one cube so that it looks like the other cube.

4. 32 teams, ranked 1 through 32, enter a basketball tournament that works as follows: the teams are randomly paired and in each pair, the team that loses is out of the competition. The remaining 16 teams are randomly paired, and so on, until there is a winner. A higher-ranked team always wins against a lower-ranked team. If the probability that the team ranked 3 (the third-best team) is one of the last four teams remaining can be written in simplest form as $\frac{m}{n}$, compute $m+n$.

5. Alice, Bob, Charlie, Diana, Emma, and Fred sit in a circle, in that order, and each roll a six-sided die. Each person looks at his or her own roll, and also looks at the roll of either the person to the right or to the left, deciding at random. Then, at the same time, Alice, Bob, Charlie, Diana, Emma and Fred each state the expected sum of the dice rolls based on the information they have. All six people say different numbers; in particular, Alice, Bob, Charlie, and Diana say 19, 22, 21, and 23, respectively. Compute the product of the dice rolls.

6. A knight is placed at the origin of the Cartesian plane. Each turn, the knight moves in an chess L-shape (2 units parallel to one axis and 1 unit parallel to the other) to one of eight possible locations, chosen at random. After 2016 such turns, what is the expected value of the square of the distance of the knight from the origin?

7. Let $a_1, a_2, a_3, \ldots$ be an infinite sequence where for all positive integers i, a_i is chosen to be a random positive integer between 1 and 2016, inclusive. Let S be the set of all positive integers k such that for all positive integers $j < k$, $a_j \neq a_k$. (So $1 \in S$; $2 \in S$ if and only if $a_1 \neq a_2$; $3 \in S$ if and only if $a_1 \neq a_3$ and $a_2 \neq a_3$; and so on.) In simplest form, let $\frac{p}{q}$ be the expected number of positive integers m such that m and $m+1$ are in S. Compute pq.

8. Katie Ledecky and Michael Phelps each participate in 7 swimming events in the Olympics (and there is no event that they both participate in). Ledecky receives g_L gold, s_L silver, and b_L bronze medals, and Phelps receives g_P gold, s_P silver, and b_P bronze medals. Ledecky notices that she performed objectively better than Phelps: for all positive real numbers $w_b < w_s < w_g$, we have
$$w_g g_L + w_s s_L + w_b b_L > w_g g_P + w_s s_P + w_b b_P.$$
Compute the number of possible 6-tuples $(g_L, s_L, b_L, g_P, s_P, b_P)$.