Geometry B Solutions

1. Let s be the radius of each of the ω_i. Then we can write $r + 2s = 1$ since the radius of the big circle is the diameter of ω_1 (for example) plus the radius of the circle in the middle. We can also draw an isosceles right triangle with vertices at the center of the middle circle, the center of ω_1, and the point of tangency of ω_1 and ω_2. This triangle gives us $r+s = s\sqrt{2}$. Subtracting the two equations, we get $s = 1 - s\sqrt{2}$. We sovle this:

$$\begin{aligned} s &= 1 - s\sqrt{2} \\ s(1+\sqrt{2}) &= 1 \\ s &= \frac{1}{1+\sqrt{2}} = \sqrt{2} - 1. \end{aligned}$$

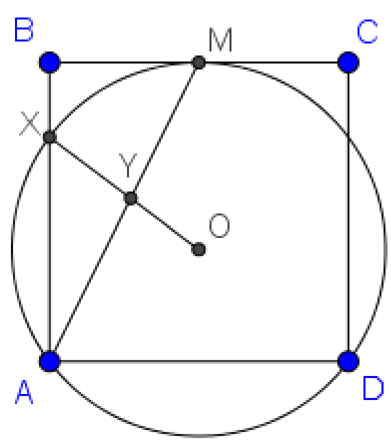
Thus, $r = 1 - 2s = 3 - 2\sqrt{2} = 3 - \sqrt{8}$. Our answer is thus $3 + 8 = \boxed{11}$.

Problem written by Eric Neyman.

2. Let T be the point of tangency and D be the intersection of AT and BC; note that AD is an altitude. Since $\angle BTC = 120°$, we find that $TD = \frac{1}{2\sqrt{3}} = \frac{AD}{3}$. Let K be the intersection of AT and PQ; since $\angle APK = \angle PTK = 60°$, $\frac{AK}{KT} = 3$. Then, $AK = \frac{3}{1+3} \cdot \left(1 - \frac{2}{3}\right) \cdot AD = \frac{AD}{2}$, so P, Q are the midpoints of AB, AC, respectively. The area of $\triangle APQ$ is then $\frac{\sqrt{3}}{16}$, so $a+b = \boxed{19}$.

Problem written by Mel Shu and Bill Huang.

3. We will reference the following picture (which we need when we use words like "horizontal" and "vertical").



Let R be the radius of ω. The distance from M to the midpoint of AD can be expressed as $R + \sqrt{R^2 - 4^2}$, and this is equal to 8. We thus have $R^2 - 4^2 = (8-R)^2$, and solving for R gives $R = 5$. The distance from O to AB is 4, so the Pythagorean theorem gives us that the vertical distance from O to X is 3. Thus, if we use a coordinate system with the natural x and y axes and origin O, then the line OX is described by the equation $y = \frac{-3}{4}x$. Meanwhile, the line AM has slope 2 and y-intercept 5 so it is described by the line $y = 2x + 5$. The intersection point Y thus has x-coordinate satisfying $2x + 5 = \frac{-3}{4}x$, i.e. $x = \frac{-20}{11}$, and this gives $y = \frac{15}{11}$. Thus, we have

$$OY = \sqrt{\left(\frac{-20}{11}\right)^2 + \left(\frac{15}{11}\right)^2} = \frac{25}{11}.$$

Our answer is thus $25 + 11 = \boxed{36}$.

Problem written by Eric Neyman.

4. The net of the curved surface of $\mathcal{C}$ is a sector of a disc. Denote by P' the point on the disc corresponding to the point P on $\mathcal{C}$. Then the shortest path along the curved surface of $\mathcal{C}$ between P_1 and P_3 corresponds to the line segment $P_1'P_3'$. By symmetry, $P_1'P_3' \perp A'P_2'$. Thus P_1' lies on the perpendicular bisector of $A'P_2'$, so $P_1'P_2' = P_1'A' = P_2'A'$ and $\triangle A'P_1'P_2'$ is equilateral (A' is the center of the disc). Then if $s = AP_1 = A'P_1'$ is the slanted height of $\mathcal{C}$, we get $2\pi r = \frac{5}{6}2\pi s \Rightarrow s = \frac{6r}{5}$. Then $\left(\frac{h}{r}\right)^2 = \frac{h^2}{r^2} = \frac{s^2-r^2}{r^2} = \frac{11}{25}$ so $a + b = 11 + 25 = \boxed{36}$.

   Problem written by Mel Shu.

5. Notice that $CFDA$ is a rhombus with side length 15. If we consider the plane of the rhombus, by symmetry we have that B and E are right above and below the center of the rhombus, which we will call O. Let $AO = x$, $CO = y$, and $BO = h$. The Pythagorean theorem gives us

$$x^2 + h^2 = AB^2 = 13^2$$
$$y^2 + h^2 = BC^2 = 14^2$$
$$x^2 + y^2 = AC^2 = 15^2.$$

   Adding the first two equations and subtracting the third one gives us $2h^1 = 140$, i.e. $h = \sqrt{70}$. Also note that $x = \sqrt{169 - 70} = 3\sqrt{11}$ and $y = \sqrt{196 - 70} = 3\sqrt{14}$. The area of the rhombus is thus $2xy = 18\sqrt{11 \cdot 14}$. Thus, the area of pyramid $ABCDF$ is

$$\frac{1}{3} \cdot 2xy \cdot h = 6\sqrt{70 \cdot 11 \cdot 14} = 84\sqrt{55}.$$

   Thus, the area of the octahedron is $168\sqrt{55}$, giving the answer $168 + 55 = \boxed{223}$.

   Problem written by Zhuo Qun Song.

6. First of all, $BF \cdot BA = BD \cdot BC \Rightarrow DC = 4$. We observe that $AE \cdot AC = AF \cdot AB = 28 \cdot 63 = 1764 = AP^2$. Thus $\angle APC = 90°$, and similarly, $CP^2 = CE \cdot CA = CD \cdot CB = 196$. Therefore $CP = \boxed{14}$.

   Problem written by Mel Shu.

7. Let $BP = 1, CP = y, AP = x$. By the law of cosines we have:

$$1^2 + x^2 + \sqrt{2} \cdot 1 \cdot x = BC^2 = (2 \cdot BP)^2 = 2^2$$
$$1^2 + y^2 + \sqrt{2} \cdot 1 \cdot y = AB^2 = AC^2 = x^2 + y^2$$

   Solving (using the fact $x > 0$) yields $x = \frac{\sqrt{7}-1}{\sqrt{2}}$ and $y = \frac{3-\sqrt{7}}{\sqrt{2}}$. Then, $\frac{CP}{AP} = \frac{x}{y} = 2 + \sqrt{7}$, so $a + b = \boxed{9}$.

   Problem written by Bill Huang.

8. Let T be the foot of the altitude from A to BX. Observe that $AT = 8$ and $BT = 6$ (integer Pythagorean triples). Then, $[DEY] = \frac{AT \cdot EY}{2}$; it remains to find $EY = AE - AY$.

   Observe that $\triangle DCX \sim \triangle ABX$, and let r be the ratio of similarity; we can write $CD = 10r$, $DX = 17r$ and $CX = 21r$. Then:

$$AE = XD \cdot \frac{CA}{CX} = 17r \cdot \frac{21r + 17}{21r}$$

Since $ABDY$ is an isosceles trapezoid, $AY = BD - 2BT = (21 + 17r) - 12 = 17r + 9$. Thus:

$$EY = AE - AY = \frac{17}{21}(21r + 17) - (17r + 9) = \frac{17^2}{21} - 9 = \frac{100}{21}$$

Therefore, $[DEY] = \frac{1}{2} \cdot 8 \cdot \frac{100}{21} = \frac{400}{21}$, so $m + n = \boxed{421}$.

Problem written by Eric Neyman.

If you believe that any of these answers is incorrect, or that a problem had multiple reasonable interpretations or was incorrectly stated, you may appeal at `tinyurl.com/pumacappeals`. All appeals must be in by 1 PM to be considered.