1. $2016 = 2^5 \cdot 3^2 \cdot 7$, so to get a cube we multiply it by $2 \cdot 3 \cdot 7^2 = 294$.

*Problem written by Eric Neyman.*

2. If $\frac{n}{s(n)}$ is a multiple of 3 then $n$ is a multiple of 3, so $s(n)$ is a multiple of 3, which means that $n$ is a multiple of $3s(n)$, so $n$ is a multiple of 9, which means that $s(n)$ is a multiple of 9, which means that $n$ is a multiple of 27. Checking $n = 27$, $n = 54$, and $n = 81$, we find that all of these values satisfy the stated condition, so the answer is $27 + 54 + 81 = 162$.

*Problem written by Eric Neyman.*

3. For each $j > 1$, we have $d_{(2,j)} = d_{(2,j-1)} + 2$, which gives $d_{(2,j)} = 2j - 1$. This means that for $j > 1$, we have

$$d_{(3,j)} = d_{(3,j-1)} + 2j - 1 + 2j - 3 = d_{(3,j-1)} + 4(j - 1).$$

Thus,

$$d_{(3,2016)} = 1 + 4 + 8 + \cdots + 4 \cdot 2015 = 1 + 4 \cdot \frac{2015 \cdot 2016}{2} \equiv 1 + 2 \cdot 15 \cdot 16 \equiv 481 \pmod{1000}.$$

*Problem written by Ryan Lee.*

4. For each $d \mid n$, pair $d$ with $\frac{n}{d}$ and observe their product is $n$. Thus, the product of all of the factors of $n$ is $n$ to the power of half the number of factors of $n$ (this also holds for perfect squares; you pair $\sqrt{n}$ with itself). Thus, $\log_n P(n)$ is equal to half the number of factors of $n$. This is an odd integer if and only if the number of factors of $n$ is divisible by 2 but not 4. Recall that if $n = p_1^{k_1} p_2^{k_2} \cdots p_m^{k_m}$ for primes $p_1$ through $p_m$, then the number of factors of $n$ is $(k_1 + 1)(k_2 + 1) \cdots (k_m + 1)$. This is divisible by 2 but not 4 if all of the $k_i$ are even except one of them, which is 1 (mod 4).

Note that $2016 = 2^5 \cdot 3^2 \cdot 7$. Let $n \mid 2016$ and write $n = 2^k 3^a 7^b$. If $k_2 \equiv 1 \pmod{4}$ then either $k_2 = 1$ or $k_2 = 5$; either way, $k_3$ is either 0 or 2 and $k_7$ is 0, giving 4 possibilities. If $k_3 \equiv 1 \pmod{4}$ then $k_3 = 1$; furthermore, $k_2$ is either 0, 2, or 4 and $k_7$ is 0, giving 3 possibilities. Finally, if $k_7 \equiv 1 \pmod{4}$ then $k_7 = 1$; furthermore, $k_2$ is either 0, 2, or 4 and $k_3$ is either 0 or 2, giving 6 possibilities. Thus, the answer is $6 + 3 + 4 = 13$.

*Problem written by Eric Neyman.*

5. Observe that if $n$ is divisible by 3 then so is $f(n)$. Thus, $n$ and $f(n)$ must not be divisible by 3. (As a consequence, $n$ cannot be prime, since then $f(n) = 3n$.) Let $g(n)$ be the smallest prime factor of $n$. Then $f(n) = n + 2g(n)$. If $g(f(n)) = g(n)$, then $f(n) = n + 2g(n)$ and $f(f(n)) = n + 4g(n)$, and it is impossible for all of $n$, $n + 2g(n)$, and $n + 4g(n)$ to not be divisible by 3. Thus, $g(f(n)) \neq g(n)$. But observe that $g(n) \mid f(n)$, so $g(f(n)) < g(n)$.

Thus, if $g(n) = 7$, then for $f(f(n))$ to not be divisible by 3, $g(f(n))$ must be 5. This means that $n + 2g(n)$ must be divisible by 5, which means that $\frac{n}{g(n)} + 2$ must be divisible by 5. Clearly $n$ cannot be $7 \cdot 3$. $n$ cannot be $7 \cdot 13$ because then $f(n) = 7 \cdot 15$ is divisible by 3. If $n = 7 \cdot 23 = 161$ then $f(n) = 7 \cdot 25 = 5 \cdot 35$, so $f(f(n)) = 5 \cdot 37$, which is not divisible by 3.

If $g(n) = 11$ and $n < 161$ then $n$ is one of $11 \cdot 11$ or $11 \cdot 13$, and in both cases $f(f(n))$ is divisible by 3. If $g(n) > 11$ then it is clear that $n$ must be greater than 161 if $f(f(n))$ is to not be divisible by 3. Therefore, the smallest $n$ is $161$.

*Problem written by Eric Neyman.*
6. The condition is equivalent to having $n(n - 1) \equiv 0 \pmod{b}$, which means that every prime power dividing $b$ divides either $n$ or $n - 1$. The Chinese remainder theorem implies that the number of different values of $n$ for which this is the case is $2$ to the power of the number of distinct primes dividing $b$, so at least four primes divide $b$. Thus, the smallest values of $b$ are $2 \cdot 3 \cdot 5 \cdot 7$ and $2 \cdot 3 \cdot 5 \cdot 11$. Adding these up, we get $2 \cdot 3 \cdot 5 \cdot 18 = 540$.

*Problem written by Eric Neyman.*

7. We have

$$\frac{\gcd(m, n)}{\text{lcm}(m, n)} = \frac{\gcd^2(m, n)}{\text{lcm}(m, n) \cdot \gcd(m, n)} = \frac{\gcd^2(m, n)}{mn} = \frac{\gcd^2(m, n)}{k}.$$ 

For each prime that goes into $k$, we can look at the minimum of the number of times it appears in $m$ and the number of times that it appears in $n$. Summing over all factors $m$ of $k$ gives us

$$(1^2 + 2^2 + 4^2 + 8^2 + 4^2 + 2^2 + 1^2)(1^2 + 3^2 + 9^2 + 9^2 + 3^2 + 1^2)(1^2 + 5^2 + 1^2)(1^2 + 7^2 + 7^2 + 1^2)(1 + 1)$$

where for instance picking the fifth summand from the first group, the second summand from the third group, and the first summand from all other groups represents $m = 2^4 \cdot 3^6 \cdot 5^1 \cdot 7^0 \cdot 53^0$. (Each summand represents the minimum of the number of times the relevant prime appears in $m$ and in $n$.) The above expression is equal to

$$\frac{106 \cdot 182 \cdot 27 \cdot 100 \cdot 2}{2^6 \cdot 3^5 \cdot 5^2 \cdot 7^3 \cdot 53} = \frac{2^5 \cdot 3^3 \cdot 5^2 \cdot 7 \cdot 13 \cdot 53}{2^6 \cdot 3^5 \cdot 5^2 \cdot 7^3 \cdot 53} = \frac{13 \cdot 2 \cdot 3^2 \cdot 7^2}{13} = 882.$$ 

so $r + s = 895$.

*Problem written by Eric Neyman.*

8. Ignore $2017^2$ — clearly it’s not among the integers with the desired property. Among the rest, for every $(a, b)$ there is exactly one integer that is $a$ modulo $2017$ and $b$ modulo $2016$, and we may treat $n^n$ as $a^b$ for the appropriate $a$ and $b$, by Fermat’s little theorem. Thus, we are looking for the number of pairs $(a, b)$ of integers with $0 \leq a < 2017$ and $0 \leq b < 2016$ such that $a^b \equiv 1 \pmod{2017}$. Now, the multiplicative group of integers modulo $2017$ is cyclic and its order is $2016$. Write this group as $g^0, g^1, \ldots, g^{2015}$. Written this way, it is clear that we are looking for the number of pairs $(k, b)$ of integers $0 \leq k, b < 2016$ such that $g^{kb} \equiv 1 \pmod{2017}$, i.e. $kb \mid 2016$. Enumerating over $b$, we note that this condition is satisfied if and only if $k$ is divisible by $\frac{2016}{\gcd(b, 2016)}$ (and there are $\gcd(b, 2016)$ such values of $k$). Thus we are looking for the sum over $0 \leq b < 2016$ of $\gcd(b, 2016)$. We note that this gcd-sum function (which we will denote $g$, i.e. we are looking for $g(2016)$) is multiplicative, because $\gcd(k, m) \gcd(k, n) = \gcd(k, mn)$ for relatively prime $m$ and $n$, and so $g(2016) = g(32)g(9)g(7)$. These three quantities can be easily evaluated, and are equal to $112, 21, \text{and } 13$, respectively, and so our answer is $112 \cdot 21 \cdot 13 = 30576$.

*Problem written by Eric Neyman.*

If you believe that any of these answers is incorrect, or that a problem had multiple reasonable interpretations or was incorrectly stated, you may appeal at tinyurl.com/pumacappeals. All appeals must be in by 1 PM to be considered.