P U M $\therefore$ C

Number Theory B

1. $2016 = 2^5 \cdot 3^2 \cdot 7$, so to get a cube we multiply it by $2 \cdot 3 \cdot 7^2 = \boxed{294}$.

   Problem written by Eric Neyman.

2. If $\frac{n}{s(n)}$ is a multiple of 3 then n is a multiple of 3, so $s(n)$ is a multiple of 3, which means that n is a multiple of $3s(n)$, so n is a multiple of 9, which means that $s(n)$ is a multiple of 9, which means that n is a multiple of 27. Checking $n = 27$, $n = 54$, and $n = 81$, we find that all of these values satisfy the stated condition, so the answer is $27 + 54 + 81 = \boxed{162}$.

   Problem written by Eric Neyman.

3. For $j > 1$, we have $d_{(2,j)} = d_{(2,j-1)} + 2$, which gives $d_{(2,j)} = 2j - 1$. This means that for $j > 1$, we have
$$d_{(3,j)} = d_{(3,j-1)} + 2j - 1 + 2j - 3 = d_{(3,j-1)} + 4(j-1).$$
   Thus,
$$d_{(3,2016)} = 1 + 4 + 8 + \cdots + 4 \cdot 2015 = 1 + 4 \cdot \frac{2015 \cdot 2016}{2} \equiv 1 + 2 \cdot 15 \cdot 16 \equiv \boxed{481} \pmod{1000}.$$

   Problem written by Ryan Lee.

4. For each $d \mid n$, pair d with $\frac{n}{d}$ and observe their product is n. Thus, the product of all of the factors of n is n to the power of half the number of factors of n (this also holds for perfect squares; you pair $\sqrt{n}$ with itself). Thus, $\log_n P(n)$ is equal to half the number of factors of n. This is and odd integer if and only if the number of factors of n is divisible by 2 but not 4. Recall that if $n = p_1^{k_1} p_2^{p_2} \dots p_m^{k_m}$ for primes p_1 through p_m, then the number of factors of n is $(k_1+1)(k_2+1)\dots(k_m+1)$. This is divisible by 2 but not 4 if all of the k_i are even except one of them, which is $1 \pmod 4$.

   Note that $2016 = 2^5 \cdot 3^2 \cdot 7$. Let $n \mid 2016$ and write $n = 2^{k_2} 3^{k_3} 7^{k_7}$. If $k_2 \equiv 1 \pmod 4$ then either $k_2 = 1$ or $k_2 = 5$; either way, k_3 is either 0 or 2 and k_7 is 0, giving 4 possibilities. If $k_3 \equiv 1 \pmod 4$ then $k_3 = 1$; furthermore, k_2 is either 0, 2, or 4 and k_7 is 0, giving 3 possibilities. Finally, if $k_7 \equiv 1 \pmod 4$ then $k_7 = 1$; furthermore, k_2 is either 0, 2, or 4 and k_3 is either 0 or 2, giving 6 possibilities. Thus, the answer is $6 + 3 + 4 = \boxed{13}$.

   Problem written by Eric Neyman.

5. Observe that if n is divisible by 3 then so is $f(n)$. Thus, n and $f(n)$ must not be divisible by 3. (As a consequence, n cannot be prime, since then $f(n) = 3n$.) Let $g(n)$ be the smallest prime factor of n. Then $f(n) = n + 2g(n)$. If $g(f(n)) = g(n)$, then $f(n) = n + 2g(n)$ and $f(f(n)) = n + 4g(n)$, and it is impossible for all of n, $n + 2g(n)$, and $n + 4g(n)$ to not be divisible by 3. Thus, $g(f(n)) \neq g(n)$. But observe that $g(n) \mid f(n)$, so $g(f(n)) < g(n)$.

   Thus, if $g(n) = 7$, then for $f(f(n))$ to not be divisible by 3, $g(f(n))$ must be 5. This means that $n + 2g(n)$ must be divisible by 5, which means that $\frac{n}{g(n)} + 2$ must be divisible by 5. Clearly n cannot be $7 \cdot 3$. n cannot be $7 \cdot 13$ because then $f(n) = 7 \cdot 15$ is divisible by 3. If $n = 7 \cdot 23 = 161$ then $f(n) = 7 \cdot 25 = 5 \cdot 35$, so $f(f(n)) = 5 \cdot 37$, which is not divisible by 3.

   If $g(n) = 11$ and $n < 161$ then n is one of $11 \cdot 11$ or $11 \cdot 13$, and in both cases $f(f(n))$ is divisible by 3. If $g(n) > 11$ then it is clear that n must be greater than 161 if $f(f(n))$ is to not be divisible by 3. Therefore, the smallest n is $\boxed{161}$.

   Problem written by Eric Neyman.

6. The condition is equivalent to having $n(n-1) \equiv 0 \pmod{b}$, which means that every prime power dividing b divides either n or $n-1$. The Chinese remainder theorem implies that the number of different values n for which this is the case is 2 to the power of the number of distinct primes dividing b, so at least four primes divide b. Thus, the smallest values of b are $2 \cdot 3 \cdot 5 \cdot 7$ and $2 \cdot 3 \cdot 5 \cdot 11$. Adding these up, we get $2 \cdot 3 \cdot 5 \cdot 18 = \boxed{540}$.

   Problem written by Eric Neyman.

7. We have
$$\frac{\gcd(m,n)}{\operatorname{lcm}(m,n)} = \frac{\gcd^2(m,n)}{\operatorname{lcm}(m,n) \cdot \gcd(m,n)} = \frac{\gcd^2(m,n)}{mn} = \frac{\gcd^2(m,n)}{k}.$$
   For each prime that goes into k, we can look at the minimum of the number of times it appears in m and the number of times that it appears in n. Summing over all factors m of k gives us
$$\frac{(1^2+2^2+4^2+8^2+4^2+2^2+1^2)(1^2+3^2+9^2+9^2+3^2+1^2)(1^2+5^2+1^2)(1^2+7^2+7^2+1^2)(1+1)}{2^6 \cdot 3^5 \cdot 5^2 \cdot 7^3 \cdot 53}$$
   where for instance picking the fifth summand from the first group, the second summand from the third group, and the first summand from all other groups represents $m = 2^4 \cdot 3^0 \cdot 5^1 \cdot 7^0 \cdot 53^0$. (Each summand represents the minimum of the number of times the relevant prime appears in m and in n.) The above expression is equal to
$$\frac{106 \cdot 182 \cdot 27 \cdot 100 \cdot 2}{2^6 \cdot 3^5 \cdot 5^2 \cdot 7^3 \cdot 53} = \frac{2^5 \cdot 3^3 \cdot 5^2 \cdot 7 \cdot 13 \cdot 53}{2^6 \cdot 3^5 \cdot 5^2 \cdot 7^3 \cdot 53} = \frac{13}{2 \cdot 3^2 \cdot 7^2} = \frac{13}{882},$$
   so $r + s = \boxed{895}$.

   Problem written by Eric Neyman.

8. Ignore 2017^2 — clearly it's not among the integers with the desired property. Among the rest, for every (a,b) there is exactly one integer that is a modulo 2017 and b modulo 2016, and we may treat n^n as a^b for the appropriate a and b, by Fermat's little theorem. Thus, we are looking for the number of pairs (a,b) of integers with $0 \le a < 2017$ and $0 \le b < 2016$ such that $a^b \equiv 1 \pmod{2017}$. Now, the multiplicative group of integers modulo 2017 is cyclic and its order is 2016. Write this group as $g^0, g^1, \ldots, g^{2015}$. Written this way, it is clear that we are looking for the number of pairs (k,b) of integers $0 \le k, b < 2016$ such that $g^{kb} \equiv 1 \pmod{2017}$, i.e. $kb \mid 2016$. Enumerating over b, we note that this condition is satisfied if and only if k is divisible by $\frac{2016}{\gcd(b,2016)}$ (and there are $\gcd(b,2016)$ such values of k). Thus we are looking for the sum over $0 \le b < 2016$ of $\gcd(b,2016)$. We note that this gcd-sum function (which we will denote g, i.e. we are looking for $g(2016)$) is multiplicative, because $\gcd(k,m)\gcd(k,n) = \gcd(k,mn)$ for relatively prime m and n, and so $g(2016) = g(32)g(9)g(7)$. These three quantities can be easily evaluated, and are equal to 112, 21, and 13, respectively, and so our answer is $112 \cdot 21 \cdot 13 = \boxed{30576}$.

   Problem written by Eric Neyman.

If you believe that any of these answers is incorrect, or that a problem had multiple reasonable interpretations or was incorrectly stated, you may appeal at `tinyurl.com/pumacappeals`. All appeals must be in by 1 PM to be considered.