P U M $\therefore$ C

Individual Finals B

1. Let $f(n)$ be the probability that, if $k \in \{1, 2, \ldots, 2n\}$ is randomly selected, then $1 + 2 + \cdots + k$ will be divisible by n. Prove that $f(n)$ is distinct for every positive integer n.

2. There are 12 candies on the table, four of which are rare candies. Chad has a friend who can tell rare candies apart from regular candies, but Chad can't. Chad's friend is allowed to take four candies from the table, but may not take any rare candies. Can his friend always take four candies in such a way that Chad will then be able to identify the four rare candies? If so, describe a strategy. If not, prove that it cannot be done.

   Note that Chad does not know anything about how the candies were selected (e.g. the order in which they were selected). However, Chad and his friend may communicate beforehand.

3. Let m, k, and c be positive integers with $k > c$, and let λ be a positive, non-integer real root of the equation $\lambda^{m+1} - k\lambda^m - c = 0$. Let $f : \mathbb{Z}^+ \to \mathbb{Z}$ be defined by $f(n) = \lfloor \lambda n \rfloor$ for all $n \in \mathbb{Z}^+$. Show that $f^{m+1}(n) \equiv cn - 1 \pmod{k}$ for all $n \in \mathbb{Z}^+$. (Here, $\mathbb{Z}^+$ denotes the set of positive integers, $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x, and $f^{m+1}(n) = f(f(\ldots f(n) \ldots))$ where f appears $m + 1$ times.)