Individual Finals B Solutions

1. $\frac{k(k+1)}{2}$ is divisible by n if and only if $k(k+1)$ is divisible by $2n$. Let $2n = p_1^{r_1}p_2^{r_2}\dots p_m^{r_m}$. Note that k and $k+1$ are relatively prime, so the only way this is possible is if k is divisible by some of the $p_i^{r_i}$ and $k+1$ is divisible by the others. There are 2^m ways to break the factors up like this, and each corresponds to exactly one k between 1 and $2n$, by the Chinese remainder theorem. Thus, $f(n) = \frac{2^m}{2n}$. It follows that $f(n_1) = f(n_2)$ only if n_1 and n_2 are off by a power of 2. But observe that m is the same for all n's that are off by a power of 2, and so $f(n)$ is distinct for all n, as desired.

   Problem written by Eric Neyman.

2. Arrange the 12 candies in a circle. We pair up the four rare candies with non-rare candies in the following manner: choose a rare candy that doesn't have a rare candy immediately to its left, pair it up with the candy immediately to its left, and remove them from the circle. Note that the ordering of the choice of rare candies doesn't matter: each rare candy is identified with the candy originally $2n-1$ candies to the left such that the $2n-2$ candies in between them have half rare, and n is minimal. Chad's friend chooses the non-rare candy in each pair. Chad can identify the four rare candies by identifying the four pairings from the chosen candies: he identifies each chosen candy with the candy $2n-1$ candies to the right such that the $2n-2$ candies in between have half chosen and half unchosen, with n minimal.

   Problem written by Zhuo Qun Song.

3. From the defining equation we obtain $\lambda = k + \frac{c}{\lambda^m} \Rightarrow \lfloor \lambda n \rfloor = kn + \left\lfloor \frac{cn}{\lambda^m} \right\rfloor$ for all $n \in \mathbb{Z}^+$, so $f(n) \equiv \left\lfloor \frac{cn}{\lambda^m} \right\rfloor \pmod{k}$. Thus $f^{m+1}(n) \equiv \left\lfloor \frac{cf^m(n)}{\lambda^m} \right\rfloor \pmod{k}$ so it suffices to show that $(cn-1)\lambda^m \le cf^m(n) < cn\lambda^m$. We note that if $\lambda = \frac{a}{b}$ is rational with $\gcd(a,b) = 1$ then $b^m\lambda^{m+1}$ is an integer, so $b = \pm 1$ and λ is an integer. Therefore λ must be irrational, so $f(n) < \lambda n \Rightarrow f^m(n) < \lambda^m n \Rightarrow cf^m(n) < cn\lambda^m$. On the other hand, $f(n) > \lambda n - 1$ so $f^m(n) > \lambda(\lambda(\dots\lambda(\lambda n - 1) - 1\dots) - 1) - 1 = \lambda^m n - \frac{\lambda^m - 1}{\lambda - 1} \Rightarrow cf^m(n) > cn\lambda^m - c\frac{\lambda^m-1}{\lambda-1}$ where $c\frac{\lambda^m-1}{\lambda-1} \le (k-1)\frac{\lambda^m-1}{\lambda-1} < \lambda^m$. The second inequality holds because $(k-1)(\lambda^m - 1) < \lambda^m(\lambda - 1) \Leftrightarrow k\lambda^m - k + 1 < \lambda^{m+1} = k\lambda^m + c$ which is true since $k + c > 1$. This completes the proof.

   Problem written by Mel Shu.