Live Round Rules and Example Test

1 Introduction

The Live Round is a test that we hope to incorporate into the full PUMaC experience in years to come. The goal is to eventually replace the Team Round with this test, or in some other way make the Live Round another competitive round for PUMaC contestants. Our motivation for introducing another round to the competition is for one simple reason: other tests such as Relays from ARML and Guts from HMMT create a type of competitive atmosphere that PUMaC has not previously offered. Math competitions are not only about doing math problems for the sake of the problems themselves. They are interesting and challenging in their own right, but there is some form of competitive spirit generated by a timed test during which one's progress can be tracked against others in real time, which makes for a different experience. This spirit is what we seek to bring to PUMaC, and with that goal in mind, we present the Live Round.

2 Format

The Live Round is *live*, meaning that scores for each team will be displayed on the screen in real time for all competitors to see. This way, teams can see how they are doing relative to all other teams at any given moment.

2.1 Problem Types

There are two types of problems:

- **Regular Problems** are more traditional math problems. They are similar to other PUMaC problems or problems from other math competitions, and they come from competition areas of Algebra, Geometry, Number Theory, and Combinatorics.
- **Exotic Problems** are problems not typically found in math competitions. These problems may range from pure estimation to calculus. These problems are intended to test your flexibility, depth of knowledge, and quickness. (Please refer to Example Set 1.3 for some examples of Exotic Problems.)

The Live Round is split into three 20-minute sections or "waves." During each wave, three sets of problems will be distributed. Problems given out during one wave are not related to problems given out during the other waves, and their answers cannot be submitted in other waves. Each wave lasts 20 minutes. (Note that our timer will count down from 60 to 0.)

Wave 1. Starts at 0 minutes. Your team is given one set (Set 1.1) of 4 **Regular Problems**.

Once your team turns in the answer sheet for Set 1.1, your team will receive Set 1.2, which also contains 4 regular problems. You must turn in your answer sheet to Set 1.1 before you can receive Set 1.2. You may only submit your answer sheet once.

Once your team turns in the answer sheet for Set 1.2, you are given Set 1.3, a set of 3 **Exotic Problems**

Note that even if your team turns in the answer sheet to Set 1.3, you are NOT given another set until Wave 1 ends.

Pause. Wave 1 ends after 20 minutes, after which there will be a 2-minute pause. This pause should give your team time to organize your strategy, and to compare yourselves to other teams. Note that any problems handed out in Wave 1 **cannot be submitted** after Wave 1 has ended.

Wave 2. Starts at 20 minutes (we will freeze the clock during the pause). The format is identical to Wave 1. You will be given one set of 4 regular problems (Set 2.1) to start. After turning in that set, you will be given another set of 4 regular problems (Set 2.2). After turning in that set, you will be given one set of 3 exotic problems (Set 2.3).

Pause. Wave 2 ends after 40 minutes, after which there will be a 2-minute pause. Problems handed out in Wave 2 **cannot be submitted** after Wave 2 has ended.

Wave 3. Starts at 40 minutes (we will freeze the clock during the pause). The format is identical to Wave 1. You will be given one set of 4 regular problems (Set 3.1) to start. After turning in that set, you will be given another set of 4 regular problems (Set 3.2). After turning in that set, you will be given one set of 3 exotic problems (Set 3.3).

End. Wave 3 ends after 60 minutes, concluding the Live Round.

3 Rules and Information

Team Test. This is a team test. Each team works on the sets together and turns in answers for each set together.

Submitting Answers. Rules for answer submission are as follows:

- Answers for a set are submitted by turning in that set's answer sheet to the team's assigned proctor.
- You may submit your answer sheet for a particular set only once.
- In any wave, a set's answers must be submitted to receive the next set. For example, Set 2.2 must be submitted to receive Set 2.3.
- You cannot turn in a set after its wave is finished. For example, you may not turn in Set 1.1 after Wave 2 has started or even during the pause between Wave 1 and Wave 2.
- You may turn in sets as early as you like. (The advantage of turning in a set early is that you have more time for later sets in that wave.)

Point Values. The points are assigned as follows:

- Problems in Set 1.1 are each worth 5 points.
- Problems in Set 1.2 are each worth 7 points.
- Problems in Set 2.1 are each worth 10 points.
- Problems in Set 2.2 are each worth 12 points.
- Problems in Set 3.1 are each worth 15 points.
- Problems in Set 3.2 are each worth 20 points.

Exotic problems are assigned point values in their problem statements.

Expected Difficulty. Each wave is more difficult than the previous wave. Within each wave, the second set is more difficult than first set. Within each set, each problem is more difficult than the previous problem. However, the difficulty is staggered so that each set should always contain easy problems and hard problems. For example, while Wave 2 is harder than Wave 1 overall, the hardest question of Wave 1 is harder than the easiest question of Wave 2.

Wave 1. Set 1.1. Regular.

1.1.1 [5] Let $f(n)$ denote the sum of the distinct positive integer divisors of n. Evaluate:

$$f(1)+f(2)+f(3)+f(4)+f(5)+f(6)+f(7)+f(8)+f(9).$$

1.1.2 [5] Sally is going shopping for stuffed tigers. She finds 5 orange, 10 white, and 2 cinnamon colored tigers. Sally decides to buy two tigers of different colors. Assuming all the tigers are distinct, in how many ways can she choose two tigers?

1.1.3 [5] How many ordered pairs (a, b) of positive integers with $1 \leq a, b \leq 10$ are there such that the geometric sequence with first term a and whose second term b has an integer as its third term?

1.1.4 [5] Ryan is messing with Brice's coin. He weights the coin such that it comes up on one side twice as frequently as the other. However, he chooses which side to weight more (either heads or tails) with equal probability. Brice flips his modified coin twice and it lands up heads both times. The probability that the coin lands up heads on the next flip can be expressed in the form $\frac{p}{q}$ for positive integers p, q satisfying $\gcd(p, q) = 1$, what is $p + q$?

Wave 1. Set 1.2. Regular.

1.2.1 [7] Imagine a regular 2015-gon with edge length 2. At each vertex, a unit circle is placed centered at that vertex and color the circle's circumference orange. Now, another unit circle S is placed inside the polygon such that it is externally tangent to two adjacent circles centered at the vertices. This circle S is allowed to roll freely in the interior of the polygon as long as it remains externally tangent to the other circles. As it rolls, the center of S draws a path in the interior of the polygon. After S rolls completely around the interior of the polygon, the total length of the path can be expressed in the form $\frac{p\pi}{q}$ for positive integers p, q satisfying $\gcd(p, q) = 1$. What is $p + q$?

1.2.2 [7] What is the smallest positive integer n such that $2^n - 1$ is a multiple of 2015?

1.2.3 [7] Charlie noticed his golden ticket was golden in two ways! In addition to being gold, it was a rectangle whose side lengths had a ratio equal to the golden ratio $\phi = \frac{1+\sqrt{5}}{2}$. He then folds the ticket so that two opposite corners (vertices connected by a diagonal) coincide and makes a sharp crease (the ticket folds just as any regular piece of paper would). The area of the resulting shape can be expressed as $a + b\phi$. What is $\frac{b}{a}$?

1.2.4 [7] Let $\sigma_1 : \mathbb{N} \to \mathbb{N}$ be a function that takes a natural number n, and returns the sum of the positive integer divisors of n. For example, $\sigma_1(6) = 1 + 2 + 3 + 6 = 12$. What is the largest number n such that $\sigma_1(n) = 1854$?

Estimation 1. Find the best rational approximation x to $\sqrt[3]{2016}$ such that $|x - \sqrt[3]{2016}|$ is as small as possible. You may either find an $x = \frac{a}{b}$ where a, b are coprime integers or find a decimal approximation.

Let C be the actual answer and A be the answer you submit. Your score will be given by $\left\lceil 10 + \frac{16.5}{0.1+e^{30|A-C|}} \right\rceil$, where $\lceil x \rceil$ denotes the smallest integer which is $\geq x$.

Calculus 1. [7] For what positive value k does the equation $\ln x = kx^2$ have exactly one solution?

Miscellaneous 1. What is the volume of the Earth in cubic meters?

Let C be the actual answer and A be the answer you submit. Your score will be given by $\max\left(0, 7 \cdot (1 - \frac{|A-C|}{C})\right)$.

Wave 2. Set 2.1. Regular.

2.1.1 [10] Triangle ABC has $\overline{AB} = 5, \overline{BC} = 4, \overline{CA} = 6$. Points D and E are on sides AB and AC, respectively, such that $\overline{AD} = \overline{AE} = \overline{BC}$. Let CD and BE intersect at F and let AF and DE intersect at G. The length of $\overline{FG}$ can be expressed in the form $\frac{a\sqrt{b}}{c}$ in simplified form. What is $a + b + c$?

2.1.2 [10] Let S be the set of integer triplets (a, b, c) with $1 \le a \le b \le c$ that satisfy $a + b + c = 77$ and:

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = \frac{1}{5}.$$

What is the value of the sum $\sum_{(a,b,c)\in S} a \cdot b \cdot c$?

2.1.3 [10] Given a rational number r that, when expressed in base-10, is a repeating, non-terminating decimal, we define $f(r)$ to be the number of digits in the decimal representation of r that are after the decimal point but before the repeating part of r. For example, $f(1.\overline{27}) = 0$ and $f(0.35\overline{2}) = 2$. What is the smallest positive integer n such that $\frac{1}{n}$, $\frac{2}{n}$, and $\frac{4}{n}$ are non-terminating decimals, where $f\left(\frac{1}{n}\right) = 3$, $f\left(\frac{2}{n}\right) = 2$, and $f\left(\frac{4}{n}\right) = 2$?

2.1.4 [10] Alice is stacking balls on the ground in three layers using two sizes of balls: small and large. All small balls are the same size, as are all large balls. For the first layer, she uses 6 identical large balls A, B, C, D, E, and F all touching the ground and so that D, E, F touch each other, A touches E and F, B touches D and F, and C touches D and E. For the second layer, she uses 3 identical small balls, G, H, and I; G touches A, E, and F, H touches B, D, and F, and I touches C, D, and E. Obviously, the small balls do not intersect the ground. Finally, for the top layer, she uses one large ball that touches D, E, F, G, H, and I. If the large balls have volume 2015, the sum of the volumes of all the balls in the pyramid can be written in the form $a\sqrt{b} + c$ for integers a, b, c where no integer square larger than 1 divides b. What is $a + b + c$?

Wave 2. Set 2.2. Regular.

2.2.1 [12] We define $\lfloor x \rfloor$ as the largest integer less than or equal to x. What is

$$\left\lfloor \frac{5^{2017015}}{5^{2015} + 7} \right\rfloor \mod 1000?$$

2.2.2 [12] Marie is painting a 4×4 grid of identical square windows. Initially, they are all orange but she wants to paint 4 of them black. How many ways can she do this up to rotation and reflection?

2.2.3 [12] Let S be the set of ordered integer pairs (x, y) such that $0 < x < y < 42$ and there exists some integer n such that $x^6 - y^6 \mid n^2 + 2015^2$. What is the sum $\sum_{(x_i,y_i)\in S} x_i y_i$?

2.2.4 [12] Let p, u, m, a, c be real numbers satisfying $5p^5 + 4u^5 + 3m^5 + 2a^5 + c^5 = 91$. What is the maximum possible value of:

$$18pumac + 2(2 + p)^2 + 23(1 + ua)^2 + 15(3 + mc)^2?$$

Wave 2. Set 2.3. Exotic.

Estimation 2. The Euclidean Algorithm on a and b is a way to find the greatest common divisor $\gcd(a, b)$. Suppose (without loss of generality) that $a > b$. On each step of the Euclidean Algorithm, we solve the equation $a = bq + r$ for integers q, r such that $0 \le r < b$, and repeat on b and r. Thus $\gcd(a, b) = \gcd(b, r)$, and we repeat. If $r = 0$, we are done. For example, $\gcd(100, 15) = \gcd(15, 10) = \gcd(10, 5) = 5$, because $100 = 15 \cdot 6 + 10$, $15 = 10 \cdot 1 + 5$, and $10 = 5 \cdot 2 + 0$. Thus, the Euclidean Algorithm here takes 3 steps. What is the largest number of steps that the Euclidean Algorithm can take on some integer inputs a, b where $0 < a, b < 10^{2016}$?

Let C be the actual answer and A be the answer you submit. If $\frac{|A-C|}{C} > \frac{1}{2}$, then your score will be 0. Otherwise, your score will be given by $\max\left\{0, \left\lceil 25 - 2\left(|A - C|/20\right)^{\frac{1}{2.2}} \right\rceil\right\}$.

Calculus 2. [12] Consider the curves with equations $x^n + y^n = 1$ for $n = 2, 4, 6, 8 \ldots$. Denote L_{2k} the length of the curve with $n = 2k$. Find $\lim\limits_{k\to\infty} L_{2k}$.

Miscellaneous 2. Suppose you play a game of poker with three players where everyone is equally skilled. The likelyhood therefore of any specific person of beating any set of other players is their proportion of chip stack. If four players have stack sizes of 5000, 10000, 20000, and 30000 chips, and the payout for finishing 4th, 3rd, 2nd, and 1st are respectively \$100, \$150, \$300, and \$500, what is the expected payout for the person currently in second place with a stack size of 20000?

Let C be the actual answer and A be the answer you submit. Your score will be given by $\max\left(0, 12 \cdot (1 - \frac{|A-C|}{C})\right)$.

Wave 3. Set 3.1. Regular.

(the questions are repeats in this example but in the Live Round all questions will be distinct)

3.1.1 [15] Triangle ABC has $\overline{AB} = 5, \overline{BC} = 4, \overline{CA} = 6$. Points D and E are on sides AB and AC, respectively, such that $\overline{AD} = \overline{AE} = \overline{BC}$. Let CD and BE intersect at F and let AF and DE intersect at G. The length of $\overline{FG}$ can be expressed in the form $\frac{a\sqrt{b}}{c}$ in simplified form. What is $a + b + c$?

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What is the value of the sum $\sum_{(a,b,c)\in S} a \cdot b \cdot c$?

3.1.3 [15] Given a rational number r that, when expressed in base-10, is a repeating, non-terminating decimal, we define $f(r)$ to be the number of digits in the decimal representation of r that are after the decimal point but before the repeating part of r. For example, $f(1.\overline{27}) = 0$ and $f(0.35\overline{2}) = 2$. What is the smallest positive integer n such that $\frac{1}{n}$, $\frac{2}{n}$, and $\frac{4}{n}$ are non-terminating decimals, where $f\left(\frac{1}{n}\right) = 3$, $f\left(\frac{2}{n}\right) = 2$, and $f\left(\frac{4}{n}\right) = 2$?

3.1.4 [15] Alice is stacking balls on the ground in three layers using two sizes of balls: small and large. All small balls are the same size, as are all large balls. For the first layer, she uses 6 identical large balls A, B, C, D, E, and F all touching the ground and so that D, E, F touch each other, A touches E and F, B touches D and F, and C touches D and E. For the second layer, she uses 3 identical small balls, G, H, and I; G touches A, E, and F, H touches B, D, and F, and I touches C, D, and E. Obviously, the small balls do not intersect the ground. Finally, for the top layer, she uses one large ball that touches D, E, F, G, H, and I. If the large balls have volume 2015, the sum of the volumes of all the balls in the pyramid can be written in the form $a\sqrt{b} + c$ for integers a, b, c where no integer square larger than 1 divides b. What is $a + b + c$? (This diagram may not have the correct scaling, but just serves to clarify the layout of the problem.)

Wave 3. Set 3.2. Regular.

(the questions are repeats in this example but in the Live Round all questions will be distinct)

3.2.1 [20] We define $\lfloor x \rfloor$ as the largest integer less than or equal to x. What is

$$\left\lfloor \frac{5^{2017015}}{5^{2015} + 7} \right\rfloor \mod 1000?$$

3.2.2 [20] Marie is painting a 4×4 grid of identical square windows. Initially, they are all orange but she wants to paint 4 of them black. How many ways can she do this up to rotation and reflection?

3.2.3 [20] Let S be the set of ordered integer pairs (x, y) such that $0 < x < y < 42$ and there exists some integer n such that $x^6 - y^6 \mid n^2 + 2015^2$. What is the sum $\sum_{(x_i,y_i)\in S} x_i y_i$?

3.2.4 [20] Let p, u, m, a, c be real numbers satisfying $5p^5 + 4u^5 + 3m^5 + 2a^5 + c^5 = 91$. What is the maximum possible value of:

$$18pumac + 2(2 + p)^2 + 23(1 + ua)^2 + 15(3 + mc)^2?$$

Wave 3. Set 3.3. Exotic.

Estimation 3. Suppose that you are standing in the middle of a 100 meter long bridge. You take a random sequence of steps either 1 meter forward or 1 meter backwards each iteration. At each step, if you are currently at meter n, you have a $n/100$ probability of going 1 meter forward, to meter $n+1$, and a $(100-n)/100$ probability of going 1 meter backward, to meter $n-1$. What is the expected value of the number of steps it takes for you to step off the bridge (i.e., to get to meter 0 or 100)?

Let C be the actual answer and A be the answer you submit. Your score will be given by $\max\left\{0, \left\lceil 25 - 25\left|\log_6\left(\frac{A-C/2}{C/2}\right)\right|^{0.8}\right\rceil\right\}$.

Calculus 3. [20] Let f be a differentiable function such that $f'(0) = 4$ and $f(0) = 3$. Compute

$$\lim_{x\to\infty}\left(\frac{f(\frac{1}{x})}{f(0)}\right)^x.$$

Miscellaneous 3. Give as many rational solutions (ie, coordinate pairs where both coordinates are rational) as you can to the equation: $y^2 + y = x^3 + x$.

You will be given $2n$ points where n is the number of correct solutions. Maximum of 20 points.