1. Note that $CC''A''A$ and $AA''B''B$ are congruent to $BB''C''C$, and their area combined with the innermost triangle is equal to the total area. Call their area $K$. Then, since the inner triangle has sides $\frac{1}{4}$ those of the larger triangle, $3K + \frac{1}{16} = 1$, so $K = \frac{5}{16}$, getting final answer $21$.

2. Let $OP = k$. Then $(60 + k)(60 - k) = \left(\frac{d}{2}\right)^2$, where $d$ is the length of the other diagonal of the kite. But $(60 + k)(60 - k) = 3600 - k^2$ is minimized and positive when $k = 59$ for $k$ an integer. Then the area of the kite is $\frac{1}{2} \cdot 120 \cdot 2\sqrt{119} = 120\sqrt{119}$, so the answer is $239$.

Problem written by Nathan Bergman

3. $\angle BDC = 90^\circ$ because it is inscribed and subtends a diameter. So, $BD$ is an altitude. Thus $[ABD] = \frac{1}{2}[ABC] = \frac{24}{1}$.  

Problem written by Zack Stier

4. The area of a parallelogram is the base times the height, so let $AD = x$, $AC = \frac{51\sqrt{55}}{2}$. Because $DAC$ is a right angle, using Pythagorean theorem, letting $CD = a$, $a^2 = x^2 + \frac{51^2 \cdot 55}{x^2}$. Clearly, $x \neq 0$, so multiplying both sides by $x^2$, $x^4 + 51^2 \cdot 55 = x^2a^2$, so $x^2a^2 - x^4 = 51^2 \cdot 55 \rightarrow x^2(a^2 - x^2) = 51^2 \cdot 55$. For $x$ to be an integer, $x^2$ must divide $51^2$, so $x = 1, 3, 17, \text{or} 51$. Testing these yields $x = 17$ and $a = 28$, so the perimeter of the parallelogram is $2(17 + 28) = 90$.

Problem written by Nathan Bergman

5. The portion of the plane inside the prism is a pentagon, which can be seen as a rectangle and a triangle. The base length of the triangle and rectangle can be calculated by drawing triangle $AFE$, which has base $6\sqrt{3}$. The height at which the rectangular portion of the pentagon stops is $\frac{2}{3}$ the height of the prism. Hence, the length of the rectangle, by the Pythagorean theorem, is $\frac{2\sqrt{106}}{3}$. The height of the triangle can also be found by the Pythagorean theorem, through legs at altitude of $BCD$ and $\frac{1}{3}$ of the height of the prism. This gives a height of $\frac{\sqrt{106}}{3}$. Hence the area of the pentagon is $6\sqrt{3} \cdot \left(\frac{2\sqrt{106}}{3} + \left(\frac{\sqrt{106}}{3}\right) / 2\right)$, or $5\sqrt{318}$ for a final answer of $323$.

Problem written by Nathan Bergman

6. Note that $\angle BPC = 180 - \angle A$. Thus, if we reflect $P$ over $BC$ to $P'$, $P'$ lies on the circumcircle of $ABC$. This means that the circumcircles of $ABC$ and $PBC$ are congruent. Their radius is $\frac{2\sqrt{3}}{3}$, so the distance between their circumcenters is $\frac{2\sqrt{3}}{3}$. Thus, $m + n + p = \frac{13}{3}$.

Problem written by Kai Zheng

7. Let $MC = x$. Then we have $MF = 11$ and $FB = 11 - x$. Consider triangle $OMC$, we have $OC^2 = OM^2 + MC^2$. Since $OM = 5$, then $OC = \sqrt{25 + x^2}$. Hence, $OA = OC = \sqrt{25 + x^2}$. Consider triangle $AHO$, we have $AH^2 = AO^2 - OH^2$, then we have $AH = \sqrt{x^2 - 96}$.

Construct a circumcircle of triangle $ABC$ and draw $AO$ intersect the circle at $T$. Therefore, we have $AB^2 = AT^2 - BT^2$. Since $AT = 2AO$, we have $AT = 2\sqrt{25 + x^2}$. Finally, we consider the triangle $ABF$ and the equation $AF^2 + FB^2 = AB^2$. Hence, $(\sqrt{x^2 - 96 + 5})^2 + (11 - x)^2 + (11 + x)^2 = 4(x^2 + 25)$. We get $x = 14$ which makes $BC = 28$.

Problem written by Nathawut Boonsiriphatthanajaroen

8. By symmetry, $\angle BEC$ is a right angle, like $\angle A$. Since BM is parallel to EC, this means $\angle EBM = \angle BEC = 90^\circ$. Thus, $MBE$ is an isoceles right triangle and, $\angle BME = \angle BEM = \angle MEC = 45^\circ$. Since $\angle KEC = \angle MEC$, $K, M, E$ are collinear, we get that triangle $KCE$ is
an isosceles right triangle as well, so $\angle KCE = 90^\circ$. Now note $\angle KZE = 45^\circ$, it cuts off an
erc, $KE$, with central angle $90^\circ$. Thus, $KZE$ is an isosceles right triangle, and $\angle CEZ = 45^\circ$.
This means that $KZ$ is perpendicular to $CE$. From before, we had that $\angle KCE = 90^\circ$, so
$KC$ is perpendicular to $CE$ as well. Thus, $K$, $C$ and $Z$ are collinear, and $ZEBK$ is an
isosceles trapezoid with base $KZ = 2KC = 2AC = 8$, and base $BE = AB = 2$, and height
$EC = AC = 4$. So its area is $20$.

*Problem written by Kai Zheng*

If you believe that any of these answers is incorrect, or that a problem had multiple reasonable
interpretations or was incorrectly stated, you may appeal at http://tinyurl.com/PUMaCappeal2017.
All appeals must be in by 1 PM to be considered.