Combinatorics A

1. Fix a particular turtle. The probability it does not get combined with another turtle when there are \( n \) total turtles is \( 1 - \frac{2}{n} \). Therefore, the probability it is not combined in 2015 seconds is

\[
\prod_{k=0}^{2014} \left(1 - \frac{2}{2017-k}\right) = \frac{2014}{2017} = \frac{2015}{2017} \cdot \frac{2016}{2017}.
\]

Since it is impossible for more than one turtle to never be combined, the desired probability is

\[
\frac{2017 \cdot 2}{(2017)(2016)} = \frac{1}{1008}.
\]

So our final answer is \(1 + \frac{1}{1008} = \frac{1009}{1008}\).

*Problem written by Matt Tyler*

2. First, we will find the sum \( N_k \) of all unremarkable numbers with at most \( k \) digits. If \( n \) is unremarkable, then we may subtract each digit from 9 and obtain a new unremarkable number, so the average of all the unremarkable numbers with at most \( k \) digits is \( \frac{10^k - 1}{2} \). The number of unremarkable numbers with at most \( k \) digits is \( 10^k - 1 \), so

\[
N_k = \left(\frac{10^k - 1}{2}\right) (10 \times 9^{k-1}).
\]

We compensate for the fact that the single-digit numbers have been excluded (since their complements are of the form 99\(_n\)), so we add back 45. Thus, the desired number is

\[
N_3 - N_2 + 45 = \frac{999}{2}(810) - \frac{99}{2}(90) + 45 = 400185.
\]

*Problem written by Matt Tyler*

3. Let \( n = 100 \). By symmetry, the probability that the first ball to roll out is black is \( \frac{1}{2} \). The probability that there are exactly \( k \) black balls and the first ball to roll out is black is \( \binom{k}{n} \frac{4}{n(n+1)(n-1)} \). Thus, the probability that there are \( k \) black balls given that the first ball is black is \( \frac{k}{2} \). (Alternatively, note that it ought to be proportional to \( k \).) Therefore, the probability there are \( k \) black balls and the second ball is black given that the first one is black is

\[
\frac{2k(k-1)}{n(n+1)(n-1)} = \binom{k}{2} \frac{4}{n(n+1)(n-1)}.
\]

The final probability is

\[
\sum_{k=2}^{n} \binom{k}{2} \frac{4}{n(n+1)(n-1)} = \binom{n+1}{3} \frac{4}{n(n+1)(n-1)} = \frac{2}{3}.
\]

Thus our answer is \(2 + 3 = 5\).

*Problem written by Matt Tyler*

4. Let the desired probability be \( p \), so that \( p \) satisfies

\[
p = \frac{1}{4}(1 + p + p^2 + p^3).
\]
This equation simplifies to
\[(p^2 + 2p - 1)(p - 1) = 0,\]
so either \(p = -1 \pm \sqrt{2}\) or \(p = 1\). Since \(-1 - \sqrt{2} < 0\), it can be rejected.

Imagine that the dice-rolling process takes place in discrete units of time. Each minute, for example, every die is rolled, and the new copies are placed to the side to be rolled during the next minute. Let \(p_k\) be the probability any given die and all its copies vanish after \(k\) minutes.

We will show by induction on \(k\) that \(p_k < \sqrt{2} - 1\), so that \(p = 1\) can also be rejected, and we can conclude \(p = -1 + \sqrt{2}\). For the base case, \(p_0 = 0 < \sqrt{2} - 1\). For the inductive step, suppose \(p_k < \sqrt{2} - 1\). Then,
\[p_{k+1} = \frac{1}{4}(1 + p_k + p_k^2 + p_k^3) < \frac{1}{4}(1 + (-1 + \sqrt{2}) + (-1 + \sqrt{2})^2 + (-1 + \sqrt{2})^3) = \sqrt{2} - 1.\]

Thus, \(p = \sqrt{2} - 1\), and the answer is \([10(\sqrt{2} + 1)] = 24\).

**Problem written by Matt Tyler**

5. Without loss of generality, say that we would like all of the strings to be returned in alphabetical order. Note that the minimum number of swaps is exactly equal to the number of pairs of letters in the original string that appear in the wrong order. (Each swap may take exactly one such pair and fix it.) We claim that if there are \(n\) distinct letters in the string, then the minimum price is
\[
\prod_{i=0}^{n-1} \sum_{j=0}^{i} 2^j = \prod_{i=1}^{n} (2^i - 1).
\]

We proceed by induction, building the string with characters that increase in alphabetical order. The base case is clear, since if there is only 1 character then there is only 1 possible string, and the cost to process it is 2\(^0\) = 1 dollar. For the inductive step, note that for every string of length \(n - 1\), the new character can be inserted in of any \(n\) spaces, and then an efficient sort is to move the new character over to the last spot and then sort the string of size \(n - 1\). Thus, using the inductive hypothesis, the total cost is
\[
\left( \prod_{i=0}^{n-2} \sum_{j=0}^{i} 2^j \right) \cdot \left( \sum_{k=0}^{i-1} 2^k \right) = \prod_{i=0}^{n-1} \sum_{j=0}^{i} 2^j
\]
as desired. The final simplification is by the finite geometric series formula. Thus, the answer for \(n = 5\) is \(1 \cdot 3 \cdot 7 \cdot 15 \cdot 31 = 9765\).

**Problem written by Jacob Wachspress**

6. Let \(p\) be the probability of staying put (we will plug in \(\frac{17}{20}\) later). Then we have
\[
E_n = \sum_{k=0}^{n} \binom{n}{k} p^k (1 - p)^{n-k-1} \cdot \frac{1}{n - k + 1} = \sum_{k=0}^{n} \frac{p^k(1 - p)^{n-k+1}\binom{n+1}{k}}{(1 - p)(n+1)}
\]
\[
= \sum_{k=0}^{n+1} \left( \frac{p^k(1 - p)^{n-k+1}\binom{n+1}{k}}{(1 - p)(n+1)} - \frac{p^{n+1}}{(1 - p)(n+1)} \right) = \frac{1 - p^{n+1}}{(1 - p)(n+1)}.
\]

Plugging in \(p = \frac{17}{20}\), we have
\[
E_n = \frac{20^{n+1} - 17^n}{3(n+1) \cdot 20^n} \approx \frac{20^n}{3(n+1) \cdot 20^n} = \frac{20}{3(n+1)},
\]
This becomes smaller than \(\frac{1}{2017}\) when \(3(n+1)\) becomes greater than \(20 \cdot 2017\), i.e. \(n = 13446\).
7. We seek to establish a recursive formula for the number $f(n)$ to place $2n$ rooks on the board such that each rook attacks exactly two other rooks; note that for this to occur each row and column must contain exactly two rooks. Viewing the rooks as vertices in a graph and edges correspond to attacking, we see that any placement $p$ can be broken down into some $c(p)$ even-length cycles of rooks attacking one another; denote by $f(n, k)$ the number of placements with $c(p) = k$. Note that given any placement $p$, we can choose $n$ non-attacking rooks. The only way to do this is for every one of the cycles, we take every other rook, yielding $2^{c(p)}$ ways to do so.

Suppose we are given $n$ rooks on the main diagonal, and wish to add $n$ more rooks, one per row and one per column (to obtain some placement $q$). Let $g(n, k)$ denote the number of ways to do this such that $c(q) = k$. Then,

$$g(n, k) = \sum_{i=2}^{n} (n - 1) \cdot \ldots \cdot (n - i + 1) \cdot g(n - i, k - 1)$$

$$= (n - 1) \cdot (g(n - 2, k - 1) + g(n - 1, k))$$

Given any placement $p$ and choice of $n$ non-attacking rooks, we can permute columns to get the $n$ non-attacking rooks on the main diagonal. Thus,

$$f(n, k) = \frac{n!}{2^k} \cdot g(n, k)$$

$$= \frac{n!}{2^k} \cdot (n - 1) \cdot (g(n - 2, k - 1) + g(n - 1, k))$$

$$= \binom{n}{2} \cdot \frac{(n - 1)!}{2^{k-1}} \cdot (g(n - 2, k - 1) + g(n - 1, k))$$

$$= \binom{n}{2} \cdot ((n - 1) \cdot f(n - 2, k - 1) + 2 \cdot f(n - 1, k))$$

We solve the recursion:

$$f(n) = \sum_{k=1}^{n} f(n, k)$$

$$= \binom{n}{2} \cdot \sum_{k=1}^{n} (n - 1) \cdot (f(n - 2, k - 1) + 2 \cdot f(n - 1, k))$$

$$= \binom{n}{2} \cdot (2 \cdot f(n - 1) + (n - 1) \cdot f(n - 2))$$

We find that $f(8) = 187530840$, so the answer is $840$.

*Problem written by Bill Huang*

8. Let Bob’s chosen string be $s$, and let the infinite sequence of bits be $S$. Let $A$ be the set of finite binary strings that do not contain $s$, and let $A(z)$ be the corresponding generating function, meaning the coefficient of $z^n$ in $A(z)$ is the number of elements of $A$ with length $n$. Similarly, let $B$ be the set of finite binary strings whose first occurrence of $s$ is at the very end, and let $B(z)$ be the corresponding generating function. Note that $A$ and $B$ are disjoint.

The coefficient of $z^n$ in $A(z/2)$ is the probability that the first $n$ bits of $S$ do not contain $s$. Therefore, if $p_n$ is the probability that the first $n - 1$ bits of $S$ do not contain $s$, but the first $n$ do, then the coefficient of $z^n$ in $A(z/2)$ is

$$\sum_{k=n+1}^{\infty} p_k,$$
and the desired expected value is
\[ \sum_{n=0}^{\infty} np_n = \sum_{n=0}^{\infty} \sum_{k=n+1}^{\infty} p_k = A(1/2). \]

Given sets \( X \) and \( Y \) of finite strings, let \( X \times Y \) be the set of strings formed by appending a string in \( Y \) to the end of a string in \( X \). Then, if \( \emptyset \) denotes the empty string, then we have
\[ A \sqcup B = \{ \emptyset \} \sqcup (A \times \{ 0 \}) \sqcup (A \times \{ 1 \}). \]

Therefore,
\[ A(z) + B(z) = 1 + 2zA(z), \]
so \( B(1/2) = 1 \).

On the other hand, the structure of \( A \times \{ s \} \) depends on the structure of \( s \). For each \( k \in \{ 0, 1, 2, 3 \} \), let \( s_k \) denote the first \( k \) digits of \( s \) (so, in particular, \( s_0 = \emptyset \)). We have the following four cases:

**Case 1:** \( s \in \{0000, 1111\} \)

We have the decomposition
\[ A \times \{ s \} = (B \times \{ s_0 \}) \sqcup (B \times \{ s_1 \}) \sqcup (B \times \{ s_2 \}), \]
so
\[ z^4 A(z) = (1 + z + z^2 + z^3)B(z), \]
so \( A(1/2) = 30 \).

**Case 2:** \( s \in \{0001, 0011, 0111, 1000, 1100, 1110\} \)

We have the decomposition
\[ A \times \{ s \} = (B \times \{ s_0 \}), \]
so
\[ z^4 A(z) = B(z), \]
so \( A(1/2) = 16 \).

**Case 3:** \( s \in \{0010, 0100, 0110, 1001, 1011, 1101\} \)

We have the decomposition
\[ A \times \{ s \} = (B \times \{ s_0 \}) \sqcup (B \times \{ s_3 \}), \]
so
\[ z^4 A(z) = (1 + z^3)B(z), \]
so \( A(1/2) = 18 \).

**Case 4:** \( s \in \{0101, 1010\} \)

We have the decomposition
\[ A \times \{ s \} = (B \times \{ s_0 \}) \sqcup (B \times \{ s_2 \}), \]
so
\[ z^4 A(z) = (1 + z^2)B(z), \]
so \( A(1/2) = 20 \).

These cases appear with probabilities \( \frac{1}{8}, \frac{3}{8}, \frac{3}{8}, \) and \( \frac{1}{8} \), respectively, so the expected value of \( A(1/2) \), and therefore the expected value of the length of the longest sequence of bits of \( S \), starting at the beginning, that does not contain \( s \), is \( \frac{30+3\cdot16+3\cdot18+20}{8} = 19 \).

**Problem written by Matt Tyler**

If you believe that any of these answers is incorrect, or that a problem had multiple reasonable interpretations or was incorrectly stated, you may appeal at http://tinyurl.com/PUMacappeal2017. All appeals must be in by 1 PM to be considered.