Individual Finals A

1. Let $\mathcal{X} = \{1, 2, \ldots, 2017\}$. Let $k$ be a positive integer. Given any $r$ such that $1 \leq r \leq k$, there exist $k$ subsets of $\mathcal{X}$ such that the union of any $r$ of them is equal to $\mathcal{X}$, but the union of any fewer than $r$ of them is not equal to $\mathcal{X}$. Find, with proof, the greatest possible value for $k$.

2. Let $a_1, a_2, a_3, \ldots$ be a monotonically decreasing sequence of positive real numbers converging to zero. Suppose that $\sum_{i=1}^{\infty} \frac{a_i}{r}$ diverges. Show that $\sum_{i=1}^{\infty} a_i^{2017}$ also diverges. You may assume in your proof that $\sum_{i=1}^{\infty} \frac{1}{r}$ converges for all real numbers $p > 1$. (A sum $\sum_{i=1}^{\infty} b_i$ of positive real numbers $b_i$ diverges if for each real number $N$ there is a positive integer $k$ such that $b_1 + b_2 + \cdots + b_k > N$.)

3. Triangle $ABC$ has incenter $I$. The line through $I$ perpendicular to $AI$ meets the circumcircle of $ABC$ at points $P$ and $Q$, where $P$ and $B$ are on the same side of $AI$. Let $X$ be the point such that $PX \parallel CI$ and $QX \parallel BI$. Show that $PB, QC$, and $IX$ intersect at a common point.