Algebra B

1. The right hand side is $x^3$, so $x^x = x^3$, so $x^x = 3$. Thus, the answer is \boxed{27}.

   *The problem had an unintended alternate solution of \boxed{1}, which was also accepted at PUMaC 2017.*

   *Problem written by Eric Neyman*

2. $x^7y^6 = (xy)^5x = (xy)^5x^2y$. $(xy)^6x$ can be formed by choosing 6 $xy$’s, 1 $x$, and 1 3, which can be done in $\binom{8}{3}(\binom{5}{1}) = 56$ ways. $(xy)^5x^2y$ can be formed by choosing 5 $x$’s, 2 $x$’s, and 1 3$y$, which can be done in $\binom{8}{2}(\binom{5}{1}) = 168$ ways. Thus the final coefficient is $56 \cdot 3 + 168 \cdot 3 = \boxed{672}$.

   *Problem written by Eric Neyman*

3. Note that $a \circ b = (a - 4)(b - 4) + 4$, so $4 \circ b = a \circ 4 = 4$ for any $a, b$. Since a 4 appears in the expression, the answer is \boxed{4}.

   *Problem written by Matt Tyler*

4. Let $w = z + \pi$. Then, we have

   $w^3 = (z + \pi)^3 = z^3 + 3z\pi(z + \pi) + \pi^3 = (2 + 2i) + 3|z|^2w + (2 - 2i) = 4 + 6w,

   so the product of all possible values of $w$ is 4. Since the real part of $z$ is $\frac{7}{5}$, the answer is \boxed{1}, which gives \boxed{3}.

   *Problem written by Matt Tyler*

5. Suppose

   $\log_{30}(a + b + c) = \log_{8}(3a) = \log_{2}(3b) = \log_{125}(3c) = x$

   for some $x$, so that $2^{2x}5^x = a + b + c$, $2^{3x} = 3a$, $3^{2x} = 3b$, and $5^{3x} = 3c$. Then,

   \[
   (a + b + c)^3 = 2^{3x}3^{2x}5^{3x} = 27abc,
   \]

   so equality holds in AM-GM, so $a = b = c = \frac{1}{3}$. Therefore, the maximum possible value of $a + 3b + 9c$ is $\frac{1}{3} + 1 + 3 = \frac{13}{3}$, which makes our final answer \boxed{16}.

   *Problem written by Matt Tyler*

6. Let $b_n = a_{n+1} - a_n$. Then, we have

   $b_n = 10a_n - (n + 1)$
   $= 10(11a_{n-1} - n) - (n + 1)$
   $= 11(10a_{n-1} - n) - 1$
   $= 11b_{n-1} - 1.$

   Therefore, if $b_1 < \frac{1}{10}$, then the sequence $b_1, b_2, \ldots$ is decreasing, and in fact goes to $-\infty$, which means that the sequence $a_1, a_2, \ldots$ does the same, and in particular becomes negative. Therefore, $b_1 \geq \frac{1}{10}$, so we have $a_2 - a_1 \geq \frac{1}{10}$, or equivalently $a_1 \geq \frac{21}{100}$. Since the sequence $a_1, a_2, \ldots$ is increasing if $a_1 = \frac{21}{100}$ (because $b_n = \frac{1}{10}$ for all $n$), our answer is \boxed{121}.

   *Problem written by Eric Neyman*

7. If $a + b = 1$, then $a^3 + b^3 = a^2 - ab + b^2$ and $a^2 + 2ab + b^2 = 1$, so

   $a^2(3 - 2a) + b^2(3 - 2b) = a^2 + 2(a^2 + b^2 - (a^3 + b^3)) + b^2 = a^2 + 2ab + b^2 = 1.$

   Therefore, by induction, $f_n(a) + f_n(b) = 1$ for all $n$, so the sum is always \boxed{2018}.

   *Problem written by Matt Tyler*
8. Let $\xi$ be a primitive thousandth root of unity, meaning 1000 is the least positive integer $n$ for which $\xi^n = 1$, so that $\{\xi^k \mid 0 \leq k < 1000\}$ are the thousand thousandth roots of unity. Then, Kenneth’s answer is

$$\sum_{k=0}^{999} \frac{1}{\xi^k - a} = \frac{-1000a^{999}}{a^{1000} - 1}$$

because $\{\xi^k - a \mid 0 \leq k < 1000\}$ is the set of roots of the polynomial $(x + a)^{1000} - 1$. Similarly, Ellen’s answer is

$$\sum_{k=0}^{999} \left( \frac{1}{\xi^k} - a \right) = -1000a,$$

because $\{\xi^k \mid 0 \leq k < 1000\}$ is the set of roots of the polynomial $x^{1000} - 1$. These answers are equal iff

$$a^{1001} - a^{999} - a = 0.$$

By Descartes’ rule of signs, this equation has one positive solution and one negative solution. Since 0 is also a solution, there are 3 possible real values of $a$.

Note: The use of the word “surprised” in the problem statement led to the interpretation that $a = 0$ would not be a possible value, as the outcome would certainly not be surprising in that instance. Therefore, 2 was also accepted for this problem.

Problem written by Matt Tyler

If you believe that any of these answers is incorrect, or that a problem had multiple reasonable interpretations or was incorrectly stated, you may appeal at http://tinyurl.com/PUMaCappeal2017. All appeals must be in by 1 PM to be considered.