Combinatorics B Solutions

1. You have four fair 6-sided dice, each numbered from 1 to 6 (inclusive). If all four dice are rolled, the probability that the product of the rolled numbers is prime can be written as \( \frac{a}{b} \), where \( a, b \) are relatively prime. What is \( a + b \)?

*Proposed by: Nathan Bergman*

**Answer:** 109

To get a prime number, you must roll 3 1’s and 1 prime number. There are 3 primes from 1 to 6 and it could have been rolled on any of the 4 dice, so there are 12 ways to achieve this out of \( 6^4 \) total possibilities. Then we have \( \frac{12}{6^4} = \frac{1}{108} \), so the answer is 109.

2. There are five dots arranged in a line from left to right. Each of the dots is colored from one of five colors so that no 3 consecutive dots are all the same color. How many ways are there to color the dots?

*Proposed by: Nathan Bergman*

**Answer:** 2800

We use complimentary counting. There are \( 5^5 = 3125 \) total ways to color the 5 dots. We now consider cases based on the maximum number of dots that are the same color. If the maximum is less than 3, we have no cases to consider, because none of these cases will have 3 consecutive dots of the same color. If the maximum is 3, then there are 5 ways to pick the color that appears 3 times, 3 ways to place these three dots so that they are consecutive, and \( 4^2 = 16 \) ways to color the last two dots, which gives \( 5 \times 3 \times 16 = 240 \). If there are 4 dots that are the same color, there are 5 ways to pick this color, 4 ways to arrange the dots so that 3 are consecutive, and 4 ways to choose the color the last dot for a total of \( 5 \times 4 \times 4 = 80 \). Lastly, if all 5 dots are the same color, there are 5 ways to pick their color. This gives a desired answer of \( 3125 - 240 - 80 - 5 = 2800 \).

3. In an election between A and B, during the counting of the votes, neither candidate was more than 2 votes ahead, and the vote ended in a tie, 6 votes to 6 votes. Two votes for the same candidate are indistinguishable. In how many orders could the votes have been counted? One possibility is AABBABBABABA.

*Proposed by: Michael Gintz*

**Answer:** 486

For some sequence \((X_1, X_2, \ldots, X_{12})\), consider the subdivision \((X_1), (X_2, X_3), (X_{10}, X_{11}), (X_{12})\). Note that after an odd number of votes, one candidate will be ahead by one point. Consider then the possibilities for one of the pairs, such as \((X_2, X_3)\). If A is ahead, the pair can be \((A, B), (B, A), \text{ or } (B, B)\), and if B is ahead, the pair can be \((A, B), (B, A), \text{ or } (A, A)\). Thus note that regardless of the previous elements, there are 3 choices for each pair. Note also that there are 2 options for the first vote, and there is only one option for the last vote (in order to tie the score). Thus there are a total of \(2 \cdot 3^5 = 486\) possibilities.

4. Let \( N \) be the number of sequences of natural numbers \( d_1, d_2, \ldots, d_{10} \) such that the following conditions hold: \( d_1 \mid d_2, \ldots, d_9 \mid d_{10} \) and \( d_{10} \mid 6^{2018} \). Evaluate the remainder when \( N \) is divided by 2017.

*Proposed by: Marko Medvedev*

**Answer:** 121

Let us write \( d_k = 2^{a_k} \cdot 3^{b_k} \). Then \( a_k \) is a nondecreasing sequence of 10 numbers, each from 0 to 2018, inclusive. Thus by sticks and stones there are \( \binom{2028}{10} \) such sequences. There are then
just as many possible sequences for \( b \), and our answer is

\[
\left( \frac{2028}{10} \right)^2 \mod 2017
\]

\[
\left( \frac{2028 \cdot 2027 \cdots 2019}{10!} \right)^2 \mod 2017
\]

\[
\left( \frac{11!}{10!} \right)^2 \mod 2017 \equiv 121 \mod 2017
\]

5. Alex starts at the origin \( O \) of a hexagonal lattice. Every second, he moves to one of the six vertices adjacent to the vertex he is currently at. If he ends up at \( X \) after 2018 moves, then let \( p \) be the probability that the shortest walk from \( O \) to \( X \) (where a valid move is from a vertex to an adjacent vertex) has length 2018. Then \( p \) can be expressed as \( a \cdot b^c \cdot n \), where \( a \), \( b \), and \( c \) are positive integers less than 10; \( a \) and \( c \) are not perfect squares; and \( m \) and \( n \) are positive integers less than 10000. Find \( a + b + c + m + n \).

*Proposed by: Eric Neyman

Answer: \( 4044 \)

Let \( H_k \) be the set of points that are accessible from \( O \) in \( k \) moves, but not \( k - 1 \) moves. Clearly \( \cup_{k=1}^n H_k \) is a hexagon of side length \( n \) centered at \( O \), so \( H_{2018} \) is the perimeter of such a hexagon of side length 2018. Say that Alex lands on some specific edge. Clearly he can only arrive there by choosing between two adjacent directions 2018 times, so there are \( 2^{2018} \) ways to arrive at one of these points. Thus there are \( 3 \cdot 2^{2019} \) ways to arrive at this hexagon, after overcounting the corners, which can only be arrived at in one way, but are counted twice, as the ends of two different edges. Thus there are a total of \( 3 \cdot 2^{2019} - 6 \) ways to arrive at the hexagon. Thus our answer is \( \left( \frac{2^{2018} - 1}{6^{2018}} \right) \), so our answer is \( 4044 \).

6. If \( a \) and \( b \) are selected uniformly from \( \{0, 1, \ldots, 511\} \) with replacement, the expected number of 1’s in the binary representation of \( a + b \) can be written in simplest form as \( \frac{m}{n} \). Compute \( m + n \).

*Proposed by: Eric Neyman

Answer: \( 6143 \)

Let \( f(n) \) be the answer for \( a \) and \( b \) selected from \( \{0, 1, \ldots, 2^n - 1\} \). Note that \( f(1) = \frac{3}{4} \). Now, consider \( f(n + 1) \). We can think of \( a \) and \( b \) as being selected from \( \{0, 1, \ldots, 2^n - 1\} \) and then with probability \( \frac{1}{2} \) adding \( 2^n \) to their sum and with probability \( \frac{1}{4} \) adding \( 2^{n+1} \). If nothing is added, the number of ones is just the number of ones in \( a + b \). If \( 2^{n+1} \) is added, it is that number plus another one in the \( 2^{n+1} \)'s place. If \( 2^n \) is added, that’s an extra 1 if and only if \( a + b \) has a 0 in its \( 2^n \)'s place, i.e. \( a + b < 2^n \). This happens with probability \( \frac{2^n(2^n+1)}{2^{2n+1}} = \frac{2^n+1}{2^n+1} \).

Thus, we have

\[
f(n+1) = f(n) + \frac{1}{4} + \frac{1}{2} \cdot \frac{2^n+1}{2^{n+1}} = f(n) + \frac{1}{2} \left( 1 + \frac{1}{2^n+1} \right).
\]

From here it is easy to find the explicit formula

\[
f(n) = \frac{n+1}{2} - \frac{1}{2^n+1}
\]

so \( f(9) = 5 - \frac{1}{1024} = \frac{5119}{1024} \), giving us an answer of \( 6143 \).
7. How many ways are there to color the 8 regions of a three-set Venn Diagram with 3 colors such that each color is used at least once? Two colorings are considered the same if one can be reached from the other by rotation and reflection.

*Proposed by: Sam Mathers*

**Answer:** 1248

We can first use Ploya’s Enumeration Theorem to get the number of ways of coloring the Venn Diagram with three colors given no restrictions. For 3 sets, A, B, and C, we have six possible symmetries, which fall into three distinct classes given by

\[(A, B, C)(AB, BC, AC)(ABC)(\emptyset) \times 2\]
\[(A, B)(C)(AB)(AC, BC)(ABC)(\emptyset) \times 3\]
\[(A)(B)(C)(AB)(AC)(BC)(ABC)(\emptyset) \times 1\]

Thus, by Ploya’s Theorem, the number of arrangements is

\[\frac{1}{6}(2 \cdot 3^4 + 3 \cdot 3^6 + 3^8) = 1485.\]

However, to account for our restriction, we must subtract away all the possible ways of coloring our Venn Diagram with just 1 or 2 of the colors. We, thus first subtract away the number of ways of 2 coloring our Venn Diagram which is

\[\frac{1}{6}(2 \cdot 2^4 + 3 \cdot 2^6 + 2^8) = 80.\]

However, we can use 2 colors in 3 different ways so we must subtract 3 \cdot 80. But, for each color, we subtracted the uniform coloring with that color twice so we must add 3 back. Thus, in total, we have 1485 - 3 \cdot 80 + 3 = 1248 ways to color the Venn Diagram with 3 colors such that every color occurs at least once.

8. Frankie the Frog starts his morning at the origin in \(\mathbb{R}^2\). He decides to go on a leisurely stroll, consisting of \(3^1 + 3^{10} + 3^{11} + 3^{100} + 3^{101} + 3^{110} + 3^{111} + 3^{1000}\) moves, starting with the 1st move. On the \(n\)th move, he hops a distance of

\[
\max\{k \in \mathbb{Z} : 3^k | n\} + 1,
\]

then turns 90° degrees counterclockwise. What is the square of the distance from his final position to the origin?

*Proposed by: Michael Gintz*

**Answer:** 496016

Let us define a *stroll* as a sequence of hops, denoted by their distances, in which Frankie begins by facing in the positive-x direction, and after each hop Frankie is to rotate 90° counterclockwise. Let \(W\) be the stroll defined in the problem. Now say \(A = a_0, \ldots, a_{1000}\) is a set of strolls, each of size \(|W|\), such that the \(n\)th hop in \(a_k\) is 1 if \(3^k | n\) and 0 otherwise.

If Frankie goes on each of the strolls in \(A\), he will end in the same location as if he had gone on the stroll \(W\), since the sum of the values of the \(n\)th hop over the elements of \(A\) is by definition the value of the \(n\)th hop in \(W\). Therefore it suffices to determine the net movement of each of the strolls in \(A\).
If we take any element of $A$ and keep track of the direction Frankie is facing on his nonzero hops, we will see that he cycles through the four directions, so it suffices to determine the remainder when the number of nonzero hops Frankie makes is divided by 4 for each element of $A$. Let us define

$$S = \{1, 10, 11, 100, 101, 110, 111, 1000\}.$$ 

Consider some $a_k$. Note first that for all $k$ we have

$$3^k > \sum_{i < k, i \in S} 3^i,$$

and each $i \geq k$ in $S$ has $3^i$ as a multiple of $3^k$. We can then use the fact that the value of $3^i/3^k \pmod{4}$ is dependent only on the parity of $i - k$ to determine the number of nonzero hops that Frankie makes in $a_k \pmod{4}$:

$$f(k) := \sum_{i \geq k, i \in S} (2i - 2k + 1) \pmod{4}.$$ 

Now we have 4 possible outcomes based on the value of $k$.

- Say $k$ is even and greater than 0. Then every value in $S$ that is at least $k$ has a corresponding value in $S$ which differs from the first by 1, with the exception of 1000. Call the smaller one $x$. Then $(2x - 2k + 1 + 2x + 2 - 2k + 1) \equiv 0 \pmod{4}$, so $f(k) \equiv 2000 - 2k + 1 \equiv 1 \pmod{4}$. The first direction Frankie faces on a nonzero hop must be the positive-$x$ direction, as $3^k \equiv 1 \pmod{4}$, so all $a_k$ with even values of $k$ which are greater than 0 are strolls whose net movement is one unit in the positive-$x$ direction.

- Say $k$ is odd but not in $S$. By the same logic as above, we have $f(k) \equiv 2000 - 2k + 1 \equiv 3 \pmod{4}$. The fourth direction Frankie faces on a nonzero hop must be the negative-$y$ direction, as Frankie always faces this direction on moves which are multiples of 4, so all $a_k$ with odd values of $k$ which are not in $S$ are strolls whose net movement is one unit in the positive-$y$ direction.

- Say $k$ is 0. The value of $f(0)$ becomes a sum of equally many ones and threes, and thus is equivalent to 0 (mod 4), so $a_0$ is a stroll with no net movement.

- Say $k$ is an odd element of $S$. Then by the same logic as before, we have $f(k) \equiv 2000 - 2k + 2 \equiv 0 \pmod{4}$, so all of these strolls have no net movement.

Since there are 500 even numbers at least 2 and at most 1000, and there are 496 positive odd numbers less than 1000 which are not in $S$, Frankie moves 500 units in the positive-$x$ direction and 496 units in the positive-$y$ direction, and our answer by the Pythagorean Theorem is $500^2 + 496^2 = 496016$.

NOTE 2: This is a bit long but if you see to think about each power separately the cases are pretty easy and I think this still belongs as 3 flames since most of the steps can be jumped over with the exception of the separation-of-powers step which isn’t so bad.

If you believe that any of these answers is incorrect, or that a problem had multiple reasonable interpretations or was incorrectly stated, you may appeal at http://tinyurl.com/PUMaCappeal2018. All appeals must be in by 1 PM to be considered.