Team

You and your team have **35 minutes** to read these instructions and complete the team round.

For each problem, you will be asked to submit two integers. The first value that you submit represents what you think the correct answer to the problem is. The second value that you submit represents **how many teams** you think will submit the correct answer. For example, consider

0. (0 points) What is $32 ÷ 2 × 4 + 3$?

The correct answer would be 67. If you think every team will get it right, you should submit the number of teams competing at PUMaC. Therefore, a viable submission for the first entry could be 67 and $n$ for the second, where $n$ is the number of teams taking the team round.

There are **72 teams** signed up to take this round at PUMaC: 45 in A division and 27 in B division. You will receive $\min(a, b)$ points for your guess, where $a$ is the number of teams that correctly answered the question, and $b$ is the number of teams you guessed would get it correct. (Note in the case that no teams answer correctly or you guess 0, you will receive 0 points.)

Each question is worth 5 points, excluding the bonus points. **These problems appear in a random order; earlier problems have not necessarily been judged to be easier!**

Good luck!

1. Let $T = \{a_1, a_2, \ldots, a_{1000}\}$, where $a_1 < a_2 < \ldots < a_{1000}$, be a uniformly randomly selected subset of $\{1, 2, \ldots, 2018\}$ with cardinality 1000. The expected value of $a_7$ can be written in reduced form as $\frac{m}{n}$. Find $m + n$.

2. Let triangle $\triangle ABC$ have $AB = 90$ and $AC = 66$. Suppose that the line $IG$ is perpendicular to side $BC$, where $I$ and $G$ are the incenter and centroid, respectively. Find the length of $BC$.

3. The value of

$$\frac{\log_3 5 \log_2 5}{\log_3 5 + \log_2 5}$$

can be expressed as $a \log_b c$, where $a$, $b$, and $c$ are positive integers, and $a + b$ is as small as possible. Find $a + 2b + 3c$.

4. For how many positive integers $n$ less than 2018 does $n^2$ have the same remainder when divided by 7, 11, and 13?

5. There exist real numbers $a$, $b$, $c$, $d$, and $e$ such that for all positive integers $n$, we have

$$\sqrt{n} = \sum_{i=0}^{n-1} \sqrt{\sqrt{ai^5 + bi^4 + ci^3 + di^2 + ei} + 1} - \sqrt{ai^5 + bi^4 + ci^3 + di^2 + ei}.$$ 

Find $a + b + c + d$.

6. Let $\tau(n)$ be the number of distinct positive divisors of $n$ (including 1 and itself). Find the sum of all positive integers $n$ satisfying $n = \tau(n)^3$.

7. Let triangle $\triangle MNP$ have side lengths $MN = 13$, $NP = 89$, and $PM = 100$. Define points $S, R, B$ as midpoints of $MN, NP, PM$ respectively. A line $\ell$ cuts lines $MN, NP, PM$ in points $I, J, A$, respectively. Find the minimum value of $(SI + RJ + BA)^2$.

8. Jackson has a $5 \times 5$ grid of squares. He places coins on the grid squares — at most one per square — so that no row, column, or diagonal has five coins. What is the maximum number of coins that he can place?
9. There are numerous sets of 17 distinct positive integers that sum to 2018, such that each
integer has the same sum of digits in base 10. Let $M$ be the maximum possible integer that
could exist in any such set. Find the sum of $M$ and the number of such sets that contain $M$.

10. For how many ordered quadruplets $(a, b, c, d)$ of positive integers such that $2 \leq a \leq b \leq c$ and
$1 \leq d \leq 418$ do we have that $bcd + abd + acd = abc + abcd$?

11. Let $\frac{a}{b}$ be a fraction such that $a$ and $b$ are positive integers and the first three digits of its
decimal expansion are 0.527. What is the smallest possible value of $a + b$?

12. In right triangle $\triangle ABC$, a square $WXYZ$ is inscribed such that vertices $W$ and $X$ lie on
hypotenuse $AB$, vertex $Y$ lies on leg $BC$, and vertex $Z$ lies on leg $CA$. Let $AY$ and $BZ$
intersect at some point $P$. If the length of each side of square $WXYZ$ is 4, the length of the
hypotenuse $AB$ is 60, and the distance between point $P$ and point $G$, where $G$ denotes the
centroid of the triangle, is $\sqrt{a}$, compute the value of $a$.

13. Consider a 10-dimensional $10 \times 10 \times \ldots \times 10$ cube consisting of $10^{10}$ unit cubes, such that one
cube $A$ is centered at the origin, and one cube $B$ is centered at $(9, 9, 9, 9, 9, 9, 9, 9, 9, 9)$. Paint
$A$ red and remove $B$, leaving an empty space. Let a move consist of taking a cube adjacent to
the empty space and placing it into the empty space, leaving the space originally contained by
the cube empty. What is the minimum number of moves required to result in a configuration
where the cube centered at $(9, 9, 9, 9, 9, 9, 9, 9, 9, 9)$ is red?

14. Find the sum of the positive integer solutions to the equation $\lfloor 3\sqrt{x} \rfloor + \lfloor 4\sqrt{x} \rfloor = 4$.

15. Aaron the Ant is somewhere on the exterior of a hollow cube of side length 2 inches, and Fred
the Flea is on the inside, at one of the vertices. At some instant, Fred flies in a straight line
towards the opposite vertex, and simultaneously Aaron begins crawling on the exterior of the
cube towards that same vertex. Fred moves at $\sqrt{3}$ inches per second and Aaron moves at $\sqrt{2}$
ingches per second. If Aaron arrives before Fred, the area of the surface on the cube from which
Aaron could have started can be written as $a\pi + \sqrt{b} + c$ where $a$, $b$, and $c$ are integers. Find
$a + b + c$.

16. Let $N$ be the number of subsets $B$ of the set $\{1, 2, \cdots, 2018\}$ such that the sum of the elements
of $B$ is congruent to 2018 modulo 2048. Find the remainder when $N$ is divided by 1000.