PUMaC 2018 Power Round:
“Life is a game I lost...”

November 18, 2018

“The Game gives you a Purpose. The Real Game is, to Find a Purpose.” — Vineet Raj Kapoor

Contents

0 Acknowledgements 2

1 Surreal Numbers (93 points) 2

1.1 Defining the Surreal Numbers (41 points) . . . . . . . . . . . . . . . . . . . 2
1.2 General Statements about Surreal Numbers (56 points) . . . . . . . . . . . 5

2 Introduction to Combinatorial Game Theory (57 points) 9

2.1 Combinatorial Game Definitions (3 points) . . . . . . . . . . . . . . . . . . . 9
2.2 $G$ (16 points) . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 9
2.3 $G$ (38 points) . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 11

3 Nim and the Sprague-Grundy Theorem (92 points) 15

3.1 Nim (18 points) . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 15
3.2 Nim Variants (41 points) . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 16
3.3 Sprague-Grundy (33 points) . . . . . . . . . . . . . . . . . . . . . . . . . . 17

4 Specific Games & Questions (240 points) 20

4.1 Toads and Frogs (75 points) . . . . . . . . . . . . . . . . . . . . . . . . . . . 20
4.2 Partizan Splittles (90 points) . . . . . . . . . . . . . . . . . . . . . . . . . . 21
4.3 Wythoff (95 points) . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 24
0 Acknowledgements

We’d like to take a moment to thank the individuals without whose contributions this Power Round would not have been nearly as successful:

• Evan Chen and Bryan Gillespie, whose thorough and thoughtful remarks helped guide this round to completion in the weeks leading up to PUMaC 2018.

A good deal of content was drawn from *Combinatorial Game Theory* by Aaron Siegel. Students looking for a more comprehensive treatment may want to look into Siegel’s text.

Much of the framework of the Surreal Numbers section was adapted from *On Numbers and Games* by John Conway. Although we created many of these problems ourselves, several were adapted straight from Conway’s book, such as Problem 1.1.9, 1.2.1, 1.2.2, 1.2.3, 1.2.4, and 1.2.5.

A lot of hard problems were taken from *Games of No Chance* and *Games of No Chance 3* by Richard Nowakowski. In particular, the entire subsection 4.1 came from Games of No Chance and the entire subsection 4.2 about Partizan Splittles came from Games of No Chance 3. 4.3 came from Aaron Siegel and from a paper by Uri Blass and Aviezri S. Fraenkel.

In addition, huge thanks go to the entire staff and volunteers of PUMaC 2018.

1 Surreal Numbers (93 points)

1.1 Defining the Surreal Numbers (41 points)

**Problem 1.1.1.** (5 points)

a) Prove for all surreal numbers $x$ that $a \in L_x$ implies $a \notin R_x$.

b) Prove that 1 is a surreal number.

c) Prove that $-1$ is a surreal number.

d) Demonstrate $-1 < 0$.

e) Demonstrate $-1 < 1$.

**Solution 1.1.1.** a) $x$ is a surreal number, so by definition each element of $L_x$ must be less than each element of $R_x$. Hence, if $a \in L_x$, then it is less than every element of $R_x$, and $a \not< a$, so $a$ cannot be in $R_x$.

b) From above, we know that $1 = \{0 | \}$. The only element of $L_1$ is 0, and 0 is less than every element of $R_1$ (which is empty), so 1 is a surreal number.

c) From above, we know that $-1 = \{| 0 \}$. $L_{-1}$ is empty, so every element in it is less than every element in $R_{-1}$ (this is vacuously true), hence $-1$ is a surreal number.

d) We have that $0 = \{ \} | -1 = \{| 0 \}$. We first show $0 \geq -1$. There is no element in $R_0$, so there is no element of $R_0$ which is $\leq -1$. And there is no element in $L_{-1}$, so there is no element of $L_{-1} \geq 0$, so $0 \geq 1$. We now show that $-1 \not< 0$. $0 \in R_{-1}$ and $0 \leq 0$, so there is an element in $R_{-1}$ which is $\geq 0$, hence $-1 \not< 0$. Because these conditions are true,
by definition, \(-1 < 0\).

e) We are given that \(0 < 1\). We have that \(1 = \{0 \mid \} \) and \(-1 = \{\mid 0\}\). We first show \(1 \geq -1\). There is no element in \(R_1\), so there is no element of \(R_1\) which is \(\leq -1\). And there is no element in \(L_{-1}\), so there is no element of \(L_{-1} \geq 1\), so \(-1 \geq 1\). We now show that \(-1 \geq 1\). 0 \(\in R_{-1}\) and 0 \(\leq 1\), so there is an element in \(R_{-1}\) which is \(\geq 1\), hence \(-1 \not\geq 1\). Because these conditions are true, by definition, \(-1 < 1\).

**Problem 1.1.2.** (5 points)

a) Demonstrate 0 = −0.

b) Demonstrate \(-1 = -(1\), proving notation is consistent for \(-1\).

c) Demonstrate 0 + 0 = 0.

d) Demonstrate 0 · 0 = 0.

e) Demonstrate 1 + 1 = 2.

**Solution 1.1.2.** a) \(0 = \{\}\\), so \(L_0 = R_0 = \{\}\\). Then by definition, \(-0 = \{-0^R \mid -0^L\}\). Because there are no elements in \(R_0\) or \(L_0\), \(-0^R\) is the empty set, and \(-0^L\) is the empty set, so \(-0 = \{\}\ = 0\).

b) \(-1\) = \(\{-1^R \mid -1^L\}\). \(L_1\) only contains 0, so \(-1^L = -0 = 0\) by part a). \(R_1\) is empty, so \(-1^R\) is the empty set. So \(-1\) = \(\{-1^R \mid -1^L\}\) = \(\{\mid 0\}\ = -1\).

c) 0 + 0 = \(\{0^L + 0, 0 + 0^L \mid 0^R + 0, 0 + 0^R\}\). Here, \(0^L\) and \(0^R\) are both the empty set, so each of these terms becomes the empty set, so \(0 + 0 = \{\}\ = 0\).

d) 0 · 0 = \(\{(0^L)0 + 0(0^L) - (0^L)(0^L), (0^R)0 + 0(0^R) - (0^R)(0^R) \mid (0^L)0 + 0(0^R) - (0^L)(0^R), (0^R)0 + 0(0^L) - (0^R)(0^L)\}\), each of which contains \(0^L\) or \(0^R\) which are both the empty set, so both sides are the empty set, and 0 · 0 = 0.

e) \(1 + 1 = \{1^L + 1, 1 + 1^L \mid 1^R + 1, 1 + 1^R\}\). \(L_1 = 0\) so \(1^L = 0\) and \(R_1\) is empty, so \(1^R\) is also a non-existing element. Then \(1^L + 1 = 0 + 1\) and \(1 + 1^L = 1 + 0\). And \(1^R + 1 = 1 + 1^R\), which is the empty set. We do not prove that 0 is the additive identity here (we prove it in later), but using this we get that \(1 + 1 = \{1\}\), which we have identified above as 2.

**Problem 1.1.3.** (5 points)

a) Find the value of \(x\) if \(x = \{2, 6 \mid \}\).

b) Find the value of \(x\) if \(x = \{-10, -4 \mid 3, 8\}\).

c) Find the value of \(x\) if \(x = \{-1, \frac{1}{2} \mid 2\}\).

d) Find the value of \(x\) if \(x = \{-1, \frac{1}{2} \mid 1, 2\}\).

e) Find the value of \(x\) if \(x = \{-\frac{5}{8}, -\frac{5}{15} \mid -\frac{1}{3}, \frac{7}{2}, \frac{729}{54}\}\).

**Solution 1.1.3.** a) We know that \(x > 6, 2,\) and \(7 > 6, 2,\) so \(x = 7\).

b) \(L_x\) and \(R_x\) are both non-empty. The earliest-born dyadic rational greater than every element of \(L_x\) and less than every element of \(R_x\) is 0, so \(x = 0\).

c) \(L_x\) and \(R_x\) are both non-empty. The earliest-born dyadic rational greater than every
element of $L_x$ and less than every element of $R_x$ is 1, so $x = 1$.

d) $L_x$ and $R_x$ are both non-empty. The earliest-born dyadic rational greater than every element of $L_x$ and less than every element of $R_x$ is $\frac{3}{4}$, so $x = \frac{3}{4}$.
e) $L_x$ and $R_x$ are both non-empty. The earliest-born dyadic rational greater than every element of $L_x$ and less than every element of $R_x$ is $-\frac{9}{32}$, so $x = -\frac{9}{32}$.

Problem 1.1.4. (6 points) Prove that a surreal number is born on a finite day if and only if it is a dyadic rationals.

Solution 1.1.4. Forward case: simple induction on days

Second case: take an arbitrary dyadic rational and provide a method to generate it from other surreal numbers born before it.

Problem 1.1.5. (2 points) Prove that if a surreal number $x$ is born on day $n$, then $L_x$ or $R_x$ contains a surreal number born on day $n - 1$.

Solution 1.1.5. Assume for contradiction the statement is false. Then because $n$ is finite, $x$ is some dyadic rational between $L_x$ and $R_x$, where each element of $L_x$ and $R_x$ are dyadic rationals born before day $n - 1$. If one of $L_x$ and $R_x$ is empty, then $x = n$ or $x = -n$, but then if $n - 1$, $-(n - 1)$ are not in their respective sets, then these are earlier born dyadic rationals that satisfy the conditions, so $x = n - 1$ or $-(n - 1)$ respectively. If $L_x$ and $R_x$ are both non-empty, then $x$ must be between some numbers of the form $\frac{a}{2^{n-2}}$, $\frac{b}{2^{n-7}}$, where $a, b$ are integers that could be multiples of two. But then there must exist some fraction with denominator $2^{n-1}$ between these fractions. This fraction is born before day $n$, so $x$ must instead be this number. Hence we have a contradiction.

Problem 1.1.6. (4 points)

a) Find the day the surreal number 2018 is born on.

b) Find the day the surreal number $-\frac{2}{5}$ is born on.

c) Find the day the surreal number $\frac{21}{8}$ is born on.

d) Find the day the surreal number $\frac{2}{3}$ is born on.

Solution 1.1.6. a) Each integer $x$ is born on day $x$, so 2018 is born on day 2018.
b) Each fraction with denominator 2 is born after the two integers closest to it have been born. $-3$ is born on day 3 and $-4$ is born on day 4, so $-\frac{7}{2}$ is born on day 5. 
c) $\frac{21}{8}$ is born from $\frac{5}{2}$ and $\frac{11}{4}$. We see that $\frac{5}{2}$ is born on day 4 (after 2 and 3 are born) and $\frac{11}{4}$ is born the next day (from $\frac{5}{2}$ and 3), which is day 5. So $\frac{21}{8}$ is born on day 6.
d) Like all other non-dyadic rationals, $\frac{2}{3}$ is born on day $\omega$.

Problem 1.1.7. (4 points) Ascertain the number of distinct surreal numbers born on or before day $n$.

Solution 1.1.7. We claim the answer is $2^{n+1} - 1$. We prove this by induction. Base case: $n = 0$. On $n = 0$, the only surreal number is 0, and $2^1 - 1 = 1$.

Inductive step: assume this is true for some $n$. On day $n + 1$, each new surreal number formed is either less than every existing one (the surreal number $-(n + 1)$), greater than
every existing one (the surreal number \((n+1)\)), or between two pre-existing surreal numbers. Then, if the number of surreal numbers born on or before day \(n\) is \(x\), then there are 2 new surreal numbers outside the interval of \(x\) and \(x-1\) new surreal numbers between them. Then on day \(n+1\), there are \(x+1\) surreal numbers born. By induction, we know there are \(2^n+1-1\) surreal numbers born on or before day \(n\), so there are \(2(2^n+1-1)+1 = 2^{n+2}-2+1 = 2^{n+2}-1\) born on or before day \(n+1\), which concludes the proof.

**Problem 1.1.8.** (6 points) Prove that \(\pi\) and \(e\) are both born on day \(\omega\).

**Solution 1.1.8.** We can write \(\pi\) and \(e\) in binary. The \(n\)th term in \(L_\pi\) will be the sum of the first \(n\) digits of \(\pi\) in binary, with a corresponding \(L_\pi\) for \(\pi\). We can also write \(4-\pi\) in binary and make the \(n\)th term of \(R_\pi\) the digits of \(4-\pi\) subtracted from 4 (with analogous definition for \(e\)). We know that neither of these numbers is born on a finite day because neither is a dyadic rational, so the earliest day they could be born is on day \(\omega\). The writing above gives a form in which both numbers are born on day \(\omega\), so we are done.

**Problem 1.1.9.** (4 points)

a) Ascertain the value of \(\{0 | \frac{1}{\omega}\}\).

b) Ascertain the value of \(\{0 | \frac{1}{\omega}, \frac{1}{4\omega}, \ldots \}\).

**Solution 1.1.9.** a) \(\{0 | \frac{1}{\omega}\} = \frac{1}{\omega}(\{0 | 1\}) = \frac{1}{\omega}(\frac{1}{2}) = \frac{1}{2\omega}\).
b) \(\{0 | \frac{1}{\omega}, \frac{1}{4\omega}, \ldots \} = \frac{1}{\omega}(\{0 | 1, \frac{1}{2}, \frac{1}{4}, \ldots \} = \frac{1}{\omega}(\frac{1}{2}) = \frac{1}{2\omega}\).

**1.2 General Statements about Surreal Numbers (56 points)**

**Problem 1.2.1.** (4 points)

a) Prove \(-(-x) = x\).

b) Prove that \(-(x+y) = -x + (-y)\).

**Solution 1.2.1.**

a) \(-(-x) = -(-x^R \mid -x^L) = \{-(x^L) \mid -(x^R)\}\). By induction, this becomes \(\{x^L \mid x^R\} = x\).

b) \(-(x+y) = -\{(x^L + y, y^L + x \mid x^R + y, x + y^R)\} = \{-x^R + (-y), -x + (-y^R) \mid -x^L + (-y), -y^L + (-x)\}\). By induction, this becomes \(\{-x^R + (-y), -x + (-y^R) \mid -x^L + (-y), -y^L + (-x)\}\), which equals \(-x + (-y)\), so we are done.

**Problem 1.2.2.** (8 points)

a) (Additive Identity) Prove that \(x + 0 = 0 + x = x\).

b) (Associative Law of Addition) Prove that \((x + y) + z = x + (y + z)\).

c) (Additive Inverse) Prove that \(x + (-x) = 0\).

**Solution 1.2.2.**
a) From the commutative property given in the Power Round, we know that $x + 0 = 0 + x$. We will show that $x + 0 = x$. $x + 0 = \{xL + 0, 0L + x \mid xR + 0 = 0R + x\}$. $0L$ and $0R$ are both empty, so this is just $\{xL + 0 \mid xR + 0\}$, which by induction is $\{xL \mid xR\} = x$, so we are done.

b) 

\[(x + y) + z = \{(x + y)L + z, (x + y) + zL \mid \ldots \} = \{(xL + y) + z, (x + yL) + z, (x + y) + zL \mid \ldots \} = \]

By induction, we have 

\[= \{xL + (y + z), x + (yL + z), x + (y + zL) \mid \ldots \} = \]

\[= x + (y + z)\]

We have an analogous approach for the right-hand side.

c) Assume for sake of contradiction $x + (-x) \not\geq 0$. Then we must have some element $a$ in $R_{x+(-x)}$ such that $a \leq 0$. So then $xR - x \leq 0$ or $x + (-xL) \leq 0$. But these are false, because by induction $xR + (-xR) \geq 0$ and $xL + (-xL) \geq 0$, so we must have $x + (-x) \geq 0$. Using the same approach to find $x + (-x) \leq 0$, we must have $x + (-x) = 0$.

**Problem 1.2.3.** (13 points)

a) Prove that $x \leq y$ if and only if $x + z \leq y + z$.

b) Show that $x - xL > 0$ and $xR - x > 0$.

c) Prove that if $x > 0$ and $y > 0$ then $x \cdot y > 0$.

d) Show that our definition of multiplication is consistent by showing that $(x \cdot y)L < (x \cdot y)R$.

**Solution 1.2.3.**

a) If $x + z \leq y + z$, then we cannot have $xR + z \leq y + z$ or $x + z \leq yL + z$, so by induction we cannot have $xR \leq y$ or $x \leq yL$, so $x \leq y$.

We prove the converse by contradiction. If $x + z \not\leq y + z$, then $zR + y \leq z + x$, $z + yR \leq z + x$, $z + y \leq zL + x$, or $z + y \leq z + xL$. And if we also have $x \leq y$, then we have $zR + y \leq z + y$, $z + yR \leq z + y$, $z + x \leq zL + x$, or $z + x \leq z + xL$, each of which is a contradiction.

Note that this implies $x + z = y + z \implies x = y$.

b) By definition, we know that $x > xL$ and $xR > x$, so we must have $x - xL > 0$ and $xR - x > 0$ by adding $-x$ or $-xL$ to both sides of the inequality.

c) Lemma: each positive $x$ has a form in which 0 is one of the $xL$, and every other $xL$ is positive.

Proof: Let $y$ be obtained from $x$ by inserting 0 as a new Left option, deleting all negative left options. It can be shown that $y$ is a surreal number that equals $x$. Let $x = \{0, xL \mid xR\}$, $y = \{0, yL \mid yR\}$. Then $L_{(x,y)}$ has 0 as a term, so it is positive.
Problem 1.2.5. (12 points) In this problem, let

a) (Multiplicative Identity) Prove that \( x \cdot 1 = x \).

b) (Commutative Property) Prove that \( x \cdot y = y \cdot x \).

c) (Negative Multiplication) Prove that \((−x)y = x(−y) = −(x \cdot y)\).

d) (Associative Property) Prove that \((x \cdot y)z = x(y \cdot z)\).

e) (Zero Product Property) Prove that \(x \cdot y = 0\) if and only if \(x = 0\) or \(y = 0\).

Solution 1.2.4.

a) \(x \cdot 1 = \{x^L1 + x^L1 - x^L1, x^R1 + x^R1 - x^R1\} \mid x^L1 + x^R1 = 1\).

b) \(x \cdot y = \{x^L1 + x^L1 - x^L1, x^R1 + x^R1 - x^R1\} \mid x^L1 + x^R1 = 1\).

By induction and commutativity of addition, this is
\[
\{y^Lx + y^Lx - y^Lx, y^Ry + y^Ry - y^Ry\} = y \cdot x
\]

c) \(-x(y) = \{−x^Ly+(−x)y^L, (−x^R)y+(−x)y^R\} \mid (−x^L)y+(−x)y^R = (−x^L)y+(−x)y^R\} = y \cdot x\)

By induction, this equals \((−xy)\). We use a similar technique to show \(x(−y) = (−xy)\), so the three are equal.

d) Expanding this out and applying induction as in the other parts of this question achieves the desired property.

e) If \(x = 0\) or \(y = 0\), \(x \cdot y = 0\) follows from the Zero Multiplication Theorem given above.

We only prove the other direction here. We prove the contrapositive: if \(x \neq 0\) and \(y \neq 0\), \(x \cdot y \neq 0\). If \(x > 0\) and \(y > 0\), \(x \cdot y > 0\) from a previous problem, so \(x \cdot y \neq 0\). If \(x, y\) both < 0, write them as \(−x\) and \(−y\) for some positive numbers \(x, y\). By previous parts of this problem \(−x(−y) = −(x)(−y) = xy\), which is greater than 0. We know commutativity of multiplication by part b), so the case where exactly one of \(x, y\) is negative can be written in the form \(−(x)y\), where \(x, y > 0\). Then we have \(−(x)y = −(xy)\). \(x > 0, y > 0\), so \(xy > 0\), so \(−(xy) < 0\). For all of these cases, \(xy \neq 0\), so we have finished the proof.

Problem 1.2.5. (12 points) In this problem, let \(y = x^{-1}\).
a) Show for all \(y^L \in L_y\) and \(y^R \in R_y\) that \(x \cdot y^L < 1 < x \cdot y^R\).

b) Show that \(y\) is a surreal number.

c) Show for all \((x \cdot y)^L \in L_{xy}\) and \((x \cdot y)^R \in R_{xy}\) that \((x \cdot y)^L < 1 < (x \cdot y)^R\).

d) Show that \(x \cdot y = 1\).

Solution 1.2.5.

a) Options of \(y\) are defined by formulas as shown:

\[
y'' = \frac{1 + (x' - x)y'}{x'}
\]

, where \(y'\) is an “earlier” option of \(y\), \(x'\) is a non-zero option of \(x\). Then we have

\[
1 - xy'' = (1 - xy')\frac{x' - x}{x'}
\]

, so \(y''\) satisfies the claim if \(y'\) does, and it does by induction (base case: 0).

b) This follows from part a)

c) The typical form of an option of \(xy\) is \(x'y + xy' - x'y' = 1 + x'(y' - y'')\), from which c) follows.

d) First observe that \(z = xy\) has a left option 0 (if \(x^L = y^L = 0\). By c), \(z^L < 1 < z^R\) for all \(z^L \in L_z\) and all \(z^R \in R_z\). Then \(z \geq 1\) because there is no element in \(R_z \leq 1\), and \(z \leq\) no element in \(L_1\) (because some element in \(L_z = 0\)). Also \(1 \geq z\) because there is no element in \(R_1 \leq z\) and 1 is \(\leq\) no element in \(L_z\), so \(z = xy = 1\).
2 Introduction to Combinatorial Game Theory (57 points)

2.1 Combinatorial Game Definitions (3 points)

Problem 2.1.1. (3 points) Prove which player can guarantee a win in Toad and Frogs played on a 1 x 6 strip with a toad in the 1st square and frogs in the 4th and 6th squares.

Solution 2.1.1. The second/right player can guarantee a win. Player 1 moves the toad to the second square no matter what. Then player 2 can move his frog from the 4th square to the third square. Then player 1 must hop the moved frog with his toad to the 4th square. Then player 2 can just keep moving his 3rd square frog to the end while player 1 won’t be able to pass the frog at the end.

2.2 \( \tilde{G} \) (16 points)

Problem 2.2.1. (10 points) Prove Definition 2.1.I and Definition 2.2.B are both equivalent definitions of short games.

Solution 2.2.1. First we prove that any game which is in \( \tilde{G} \) is short and loopfree. Let \( G \in \tilde{G} \). Now \( G \in \tilde{G}_n \) for some \( n \). Otherwise it isn’t in \( \tilde{G} \). But now because of the definition of \( \tilde{G}_n \), we have that \( G^L \in \tilde{G}_{n-1} \) and also \( G^R \in \tilde{G}_{n-1} \). Now let any player play a move and let the new game be \( G' \). We can infer now \( G' \in \tilde{G}_k \), where \( k < n \). So after at most \( n \) moves our position will be in \( G \in \tilde{G}_0 \), when the game ends. (More formally, the number \( n \) in the process of playing the game is a strictly decreasing and at some point it hits 0). That’s why it’s finite. Now suppose that \( G \) is not loopfree and take the minimal \( n \) such that \( G \in \tilde{G}_n \). Now let the players play the sequence of moves such that they get to \( G \) again. But now as we saw earlier the number \( n \) decreases with each turn. Let’s say the players return to \( G \) after \( k \) moves. Now we get \( G \in \tilde{G}_m \), where \( m \leq n-k < n \), hence a contradiction with the minimality of \( n \).

Now take \( G \) which is short and loopfree. Construct a ”game graph” for \( G \) where every position is a node and there is a directed edge between \( A \) and \( B \) if \( B \) can be reached from \( A \) in 1 move. From the finiteness of \( G \) we see that this graph is finite. Since \( G \) is loopfree the graph has no cycles. Now 0 is in the graph.

From every position there is a path to 0. This is true because the game is finite and loopfree. Let \( P \) be a position in \( G \) and \( n \) the number of positions that can be reached from \( G \). Now 0 is one of those positions for \( G \), because the game must always end. Now, we make as many as possible moves from \( P \), and at most \( n \). Since \( G \) is loopfree we will visit all the positions that are possible after \( P \), so we must also visit 0.

We induct on the nodes of \( G \). We claim that for every \( v \) in a set \( S \), \( v \) belongs to \( \tilde{G}_n \) for some \( n \). For 0 this certainly holds.

Now we claim that that there is a node such that we can only get to \( S \) from it. Since 0 is in \( S \) for every node in \( V - S \) there is a path to \( S \). Now take \( v \) and take a path from it to 0, and in that path take \( v' \), the last node that is in \( V - S \), and denote the path from \( v \) to \( v' \) as \( P_v \). Suppose that \( (v', w) \in E \), where \( w \in V - S \). Now repeat the process for \( w \) and select \( w' \). Now make a path \( P_1 = P_v \cup P_w \). Now if we repeat this process we will get from our assumption an arbitrary long path in \( V - S \), but from the finiteness of \( V - S \) this path contains a cycle, which contradicts the fact that \( G \) is loopfree.
Hence our assumption was wrong, there exists \( u \) such that \( u \) is connected only to nodes in \( S \). Now let \( n \) be the maximal \( i \) such that \( \forall v \in S, v \in \tilde{G}_i \). Now by the definition \( u \in \tilde{G}_{n+1} \). So we increased the size of \( S \). Eventually, \( S = V \), and \( G \in \tilde{G}_n \) for some \( n \).

**Problem 2.2.2.** (2 points)

a) Explain why the 0 game is a 2nd player win.

b) Explain why the * game is a 1st player win.

**Solution 2.2.2.**

a) The first player has no options, so he cannot move at all. Hence the second player wins.

b) The first player has as his only option the 0 game. Then from the 0 game, the second player of that game, who is the first player of the * game, will win as shown above.

**Problem 2.2.3.** (4 points) (Fundamental Theorem) Let \( G \) be short and assume normal play. Prove that either the left player can force a win playing first or else the right player can force a win playing second, but not both.

**Solution 2.2.3.** Consider a typical left option \( G^L \). Since \( G \) is short, \( G^L \) has strictly fewer sub-positions than \( G \). Therefore we may assume by induction (and symmetry) that either right can force a win playing first on \( G^L \), or else left can force a win playing second.

If right can win all such \( G^L \) playing first, then certainly he can win \( G \) playing second regardless of left’s opening move. Conversely, if left can win any such \( G^L \) playing second, then he can win \( G \) by moving to it. Exactly one of these two possibilities must hold.

**Problem 2.2.4.** (4 points)

a) Is Hackenbush a game of normal play or misère play?

b) Is Hackenbush an impartial game?

c) Prove that Hackenbush is a short game.

**Solution 2.2.4.**

a) Hackenbush is a game of normal play because the last person who removes a line from the board wins the game.

b) Hackenbush is not an impartial game because one player can remove one set of line segments, while the other player can remove another set of line segments.

c) To show Hackenbush is short, we show that it is finite and loopfree. Hackenbush is finite because the board contains a finite amount of line segments, and each move removes at least one line segment, so if the original board has \( n \) segments, the game will be over in at most \( n \) turns. Hackenbush is loopfree because in order to return to a previous board state with \( n \) line segments, because each turn removes at least one segment, we must add in some amount of line segments on a turn, which is impossible by the rules of the game. Because Hackenbush is finite and loopfree, it is short.
Problem 2.2.5. (2 points)

a) Prove for any short game $G$ we have $-(-G) = G$.

b) Evaluate $\ast + \ast$.

Solution 2.2.5.

a) $-(-G) = -\{-G^R \mid -G^L\} = \{-(-G^L) \mid -(-G^R)\}$. By induction, this becomes $\{G^L \mid G^R\} = G$.

b) $\ast + \ast = \{0^L + 0, 0^L + 0 \mid 0^R + 0, 0 + 0^R\} = \{\}\} = 0$.

2.3 $\mathbb{G}$ (38 points)

Problem 2.3.1. (3 points) Prove that $\equiv$ (in the context of games) is an equivalence relation.

Solution 2.3.1. $G \equiv H$ if and only if $H \equiv G$ as $o(G + X) = o(H + X)$ for all $X$ trivially.

Problem 2.3.2. (6 points) Let $G$ be an impartial game. Prove $G$ is equivalent to 0 if and only if $G$ is a $P$-position.

Solution 2.3.2. The forward direction is easy. If $G = 0$ then for the short game $X = 0$ we have $o(G) = o(0)$ for $G$ is a $P$-position.

Problem 2.3.3. (5 points) Prove $\mathbb{G}$ is an abelian group.

Solution 2.3.3. $\mathbb{G}$ is easily closed from definition.

Problem 2.3.4. (2 points) Demonstrate the game value of Hackenbush Game 1 below.
Solution 2.3.4. We add the values of the two lines together. The value of the blue line is 2, and the value of the red line is $-3$. $2 - 3 = -1$.

Problem 2.3.5. (2 points) Demonstrate the game value of Hackenbush Game 2 below.

Solution 2.3.5. There are just 4 blue lines; Left player will remove them 1 at a time, so the game value is 4.

Problem 2.3.6. (4 points) Demonstrate the game value of Hackenbush Game 3 below.
Solution 2.3.6. The game with the bottom blue line and the two red lines above it has value \( \frac{1}{4} \). The game with the two blue lines and one of the two lower red lines has value 1, so the game with the bottom four segments has game value \( \left\{ \frac{1}{4} \mid 1 \right\} = \frac{1}{2} \). Then the game with one removed red line has value \( \left\{ \frac{1}{4} \mid \frac{1}{2} \right\} = \frac{3}{8} \). Left player can either make the game 0 (remove the bottom line) or \( \frac{1}{4} \) (remove the top line), so the optimal move is to remove the top line. Right will want to remove one of the top red lines, which gives a game value of \( \frac{3}{8} \), so the game value is \( \left\{ \frac{1}{4} \mid \frac{3}{8} \right\} = \frac{5}{16} \).

Problem 2.3.7. (3 points) Demonstrate the game value of Hackenbush Game 4 below.

\[
\text{Hackenbush Game 4}
\]

Solution 2.3.7. Left line: \( 1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \frac{1}{16} = \frac{11}{16} \). Right line: \( \left\{ \frac{3}{2} \mid 2 \right\} = \frac{7}{4} \). Then the total value is \( \frac{11}{16} - \frac{7}{4} = -\frac{17}{16} \).

Problem 2.3.8. (5 points) Demonstrate the game value of Hackenbush Game 5 below.

\[
\text{Hackenbush Game 5}
\]

Solution 2.3.8. Working backwards, the game with just the two blue “legs” has value 2, and the game with those legs and the “spine” has value 1. Adding an arm gives the game
with value \( \{1 \mid 2\} = \frac{3}{2} \). The game with both arms has value \( \{\frac{3}{2} \mid 2\} = \frac{7}{4} \). The game with two legs and two red lines above has value \( \{\frac{3}{2} \mid 1\} = \frac{7}{4} \). The game with 3 emerging red lines has value \( \{\frac{1}{2} \mid \frac{1}{2}\} = \frac{1}{2} \), and the game with all the lines except the arms has value \( \{\frac{1}{16} \mid \frac{1}{2}\} = \frac{1}{8} \). The game with one arm and two red lines has value \( \{\frac{1}{2} \mid \frac{3}{2}\} = 1 \), so the game with both arms and two red lines has value \( \{1 \mid \frac{7}{4}\} = \frac{3}{2} \). The game with one arm and three red lines has value \( \{\frac{1}{4} \mid 1\} = \frac{1}{2} \), so the corresponding game with two arms has value \( \{\frac{1}{4} \mid \frac{3}{4}\} = \frac{3}{4} \). Lastly, the game with all lines except for one arm has value \( \{\frac{1}{8} \mid \frac{3}{4}\} = \frac{1}{4} \), so the game with all lines has value \( \{\frac{1}{16} \mid \frac{3}{4}\} = \frac{1}{8} \).

**Problem 2.3.9.** (8 points) Demonstrate the game value of Hackenbush Game 6 below.

![Hackenbush Game 6](image)

**Solution 2.3.9.** We go letter by letter then add up all the individual game values. The \( U \) has value 4 because it consists of 4 blue lines, and the \( C \) has value \(-2\) because it consists of 2 red lines. Using the same techniques from the previous parts, the \( P \) has value \( \{-\frac{3}{2} \mid -\frac{3}{4}\} = -1 \), the \( M \) has value \( \{-3 \mid -\frac{3}{2}\} = -2 \), and the \( A \) has value \( \{\frac{3}{4} \mid 2\} = 1 \). Adding all these together gives a final answer of \( 4 - 2 - 1 - 2 + 1 = 0 \).
3 Nim and the Sprague-Grundy Theorem (92 points)

3.1 Nim (18 points)

Problem 3.1.1. (4 points)

a) Find the nim-value of a game of nim with piles of token size 5, 6, 2, 9, and 2018.

b) Find the nim-value of a game of nim with 2018 piles of token size 2018.

Solution 3.1.1.

a) 

$$5 \oplus 6 \oplus 2 \oplus 9 \oplus 2018 = 101 \oplus 110 \oplus 10 \oplus 1001 \oplus 1111100010 = 1111101010 = 2026$$

b) Note 2018 \oplus 2018 = 0 so the nim value is 1009 xor’s of 0, which is 0.

Problem 3.1.2. (6 points) (Bouton’s Theorem) Let G be a nim position. Prove that if G is a zero position, then every move from G leads to a nonzero position. Prove that if G is a nonzero position, then there exists a move from G to a zero position.

Solution 3.1.2.

First suppose $a_1 \oplus a_2 \oplus \cdots \oplus a_k = 0$ and consider a typical move from $a_1$ to $a'_1$. Necessarily $a_1 \neq a'_1$, so

$$a'_1 \oplus a_2 \oplus \cdots \oplus a_k \neq a_1 \oplus a_2 \oplus \cdots \oplus a_k = 0$$

Conversely, suppose $x = a_1 \oplus \cdots \oplus a_k \neq 0$. Consider the most-significant bit, say the $j^{th}$ bit, of the binary representation of $x$. At least one of the $a_i$ must have its $j^{th}$ bit equal to 1; assume without loss of generality that it is $a_1$. Put $a'_1 = x \oplus a_1$. Necessarily $a_1 < a'_1$, since its $j^{th}$ bit is equal to 0 and it agrees with $a_1$ on all higher-order bits. So there is a move available from $a_1$ to $a'_1$, and

$$a'_1 \oplus \cdots \oplus a_k = x \oplus a_1 \oplus \cdots \oplus a_k = x \oplus x = 0$$

Problem 3.1.3. (4 points) Find with proof the $N$-positions and $P$-positions of Nim.

Solution 3.1.3. From Bouton’s theorem we have if $G$ is a zero position, second player can guarantee a win by reverting each of his opponent’s moves to a new zero position. Since every move reduces the total number of tokens in play, the game will eventually end and second player will have the last move.

If $G$ is not a zero position, then first player can guarantee a win, simply by moving to any zero position.

Problem 3.1.4. (4 points) Prove for all $a, b \in \mathbb{N}$ we have $*$a + *b = *(a + b).

Solution 3.1.4. Note both $*$a + *b and *(a + b) have the same nim value, and hence are in the same outcome class when added to a short game $X$. 

3.2 Nim Variants (41 points)

**Problem 3.2.1.** (5 points) In a game of misère nim, find with proof the $N$-positions and $P$-positions (recall the definition of misère from a previous section).

**Solution 3.2.1.** Winning Strategy:
Play exactly like you would in normal play until your opponent leaves one pile of size greater than one.
At this point, reduce this pile to size 1 or 0, whichever leaves an odd number of piles with only one object.
Why this works:
Optimal play from normal Nim will never leave you with exactly one pile of size greater than one (because this leaves a losing position).
Your opponent can’t move from two piles of size greater than one to no piles of size greater than one.
So eventually your opponent must make a move that leaves only one pile of size greater than one.

**Problem 3.2.2.** (8 points) Find with proof the $N$-positions and $P$-positions of normal play Triple Nim.

**Solution 3.2.2.** The winning strategy is as follows: Like in ordinary Nim, one considers the binary representation of the heap sizes. In ordinary Nim one forms the XOR-sum (or sum modulo 2) of each binary digit, and the winning strategy is to make each XOR sum zero. In the generalization to $r$-Nim, one forms the sum of each binary digit modulo $r + 1$. Again the winning strategy is to move such that this sum is zero for every digit. Indeed, the value thus computed is zero for the final position, and given a configuration of heaps for which this value is zero, any change of at most $r$ heaps will make the value non-zero. Conversely, given a configuration with non-zero value, one can always take from at most $r$ heaps, carefully chosen, so that the value will become zero.

**Problem 3.2.3.** (4 points) Find with proof the $N$-positions and $P$-positions of misère play Triple Nim.

**Solution 3.2.3.** Combination of misère nim and normal Triple Nim.

**Problem 3.2.4.** (5 points) Find with proof the $N$-positions and $P$-positions of normal play $(n, r)$-Nim.

**Solution 3.2.4.** The $P$-positions are those positions that are multiples of $r + 1$. $N$-positions are not multiples of $r + 1$. If the starting position is a multiple of $r + 1$ then the first player will have to move to a non-multiple of $r + 1$, and then the second player will move to a multiple of $r + 1$. The reverse will apply in the non-multiple of $r + 1$ case.

**Problem 3.2.5.** (5 points) Find with proof the $N$-positions and $P$-positions of misère play $(n, r)$-Nim.

**Solution 3.2.5.** The $P$-positions are those positions that are 1 mod $k + 1$. Then $N$-positions are the other positions. If the position is 1 mod $k + 1$ then the second player
will always move the pile back to \( 1 \mod k + 1 \) after the first player. Then the first player will be forced to take 1 coin at the end as a final move and lose. If the position is not \( 1 \mod k + 1 \) then the reverse will apply.

**Problem 3.2.6.** (12 points) Find with proof the \( N \)-positions and \( P \)-positions of normal play Tiger Nim.

**Solution 3.2.6.** The optimal strategy in Tiger nim can be described in terms of the “quota” \( q \) (the maximum number of coins that can currently be removed: all but one on the first move, and up to twice the previous move after that) and the Zeckendorf representation of the current number of coins as a sum of non-consecutive Fibonacci numbers. A given position is a losing position (for the player who is about to move) when \( q \) is less than the smallest Fibonacci number in this representation, and a winning position otherwise. In a winning position, it is always a winning move to remove all the coins (if this is allowed) or otherwise to remove a number of coins equal to the smallest Fibonacci number in the Zeckendorf representation. When this is possible, the opposing player will necessarily be faced with a losing position, because the new quota will be smaller than the smallest Fibonacci number in the Zeckendorf representation of the remaining number of coins. From a losing position, any move will lead to a winning position. In particular, when there is a Fibonacci number of coins in the starting pile, the position is losing for the first player (and winning for the second player). However, when the starting number of coins is not a Fibonacci number, the first player can always win with optimal play.

**Problem 3.2.7.** (2 points) Find with proof the \( N \)-positions and \( P \)-positions of misère play Tiger Nim.

**Solution 3.2.7.** Every game is a \( N \)-position as the first player can just remove all but one of the coins, and the second player will have to remove the last coin and lose.

### 3.3 Sprague-Grundy (33 points)

**Problem 3.3.1.** (10 points) For every short impartial games \( G, G' \), we have \( G = G' \) if and only if \( G + G' \) is a \( P \)-position.

**Solution 3.3.1.** First we prove a quick lemma. We prove for every position \( G \) and every \( P \)-position \( A \), the equivalence \( G = A + G \).

Suppose that \( G + H \) is a \( P \)-position. Then the previous player has a winning strategy for \( A + G + H \): respond to moves in \( A \) according to their winning strategy for \( A \) (which exists by virtue of \( A \) being a \( P \)-position), and respond to moves in \( G + H \) according to their winning strategy for \( G + H \) (which exists for analogous reason). So \( A + G + H \) must also be a \( P \)-position.

On the other hand, if \( G + H \) is an \( N \)-position, then the next player has a winning strategy: choose a \( P \)-position from among the \( G + H \) options, putting their opponent in the case above. Thus, in this case, \( A + G + H \) must be a \( N \)-position, just like \( G + H \).

As these are the only two cases, the lemma holds.

So now, in the forward direction, suppose that \( G = G' \). Applying the definition of equivalence with \( X = G \), we find that \( G' + G \) (which is equal to \( G + G' \) by commutativity of
addition) is in the same outcome class as $G + G$. But $G + G$ must be a $P$-position: for every move made in one copy of $G$, the previous player can respond with the same move in the other copy, and so always make the last move.

In the reverse direction, since $A = G + G'$ is a $P$-position by hypothesis, it follows from the first lemma, $G = G + A$, that $G = G + (G + G')$. Similarly, since $B = G + G$ is also a $P$-position, it follows from the first lemma in the form $G' = G' + B$ that $G' = G' + (G + G)$. By associativity and commutativity, the right-hand sides of these results are equal. Furthermore, $=$ is an equivalence relation. Via the transitivity of $=$, we can conclude that $G = G'$.

**Problem 3.3.2.** (8 points) Let $a_1, a_2, \ldots, a_j \in \mathbb{N}$, and suppose that $G = \{a_1, a_2, \ldots, a_k | a_1, a_2, \ldots, a_k\}$. Prove that $G = *m$, where $m = \text{mex}\{a_1, a_2, \ldots, a_n\}$.

**Solution 3.3.2.** Now let us show that $G + *m$ is a $P$-position, which, using problem 3.3.1, means that $G' = *m$. We do so by giving an explicit strategy for the previous player.

Suppose that $G$ and $*m$ are empty. Then $G + *m$ is the null set, clearly a $P$-position.

Or consider the case that the next player moves in the component $*m$ to the option $*m'$ where $m' < m$. Because $m$ was the minimum excluded number, the previous player can move in $G$ to $*m'$. And, as shown before, any position plus itself is a $P$-position.

Finally, suppose instead that the next player moves in the component $G$ to the option $*n_i$. If $n_i < m$ then the previous player moves in $*m$ to $*n_i$; otherwise, if $n_i > m$, the previous player moves in $*n_i$ to $*m$; in either case the result is a position plus itself. (It is not possible that $n_i = m$ because $m$ was defined to be different from all the $n_i$.)

**Problem 3.3.3.** (6 points) (Sprague-Grundy Theorem) Every short impartial game under the normal play convention is equivalent to a nimber.

**Solution 3.3.3.** Consider a short game $G = \{G_1, G_2, \ldots, G_k | G_1, G_2, \ldots, G_k\}$. By induction each of the options equal a nimber. So $G' = \{n_1, n_2, \ldots, n_k | n_1, n_2, \ldots, n_k\} = *m = \text{mex}\{n_1, n_2, \ldots, n_k\}$ from above problem 3.3.2. We prove $G = G'$. If $k$ is zero, the claim is trivially true. Otherwise, consider $G + G'$. If the next player makes a move to $G_i$ in $G$, then the previous player can move to $*n_i$ in $G'$, and conversely if the next player makes a move in $G'$. After this, the position is a $P$-position by problem 3.3.1 forward implication. Therefore, $G + G'$ is a $P$-position, and, citing problem 3.3.1 reverse implication, $G = G'$.

Hence by transitivity $G = *m$.

**Problem 3.3.4.** (9 points) Find the $G$-values for a game of Dawson’S Kayles consisting of one row of $n$ boxes for each integer $n$ from 0 to 17.

**Solution 3.3.4.**
<table>
<thead>
<tr>
<th>$n$</th>
<th>$G$-value</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>3</td>
<td>1</td>
</tr>
<tr>
<td>4</td>
<td>2</td>
</tr>
<tr>
<td>5</td>
<td>0</td>
</tr>
<tr>
<td>6</td>
<td>3</td>
</tr>
<tr>
<td>7</td>
<td>1</td>
</tr>
<tr>
<td>8</td>
<td>1</td>
</tr>
<tr>
<td>9</td>
<td>0</td>
</tr>
<tr>
<td>10</td>
<td>3</td>
</tr>
<tr>
<td>11</td>
<td>3</td>
</tr>
<tr>
<td>12</td>
<td>2</td>
</tr>
<tr>
<td>13</td>
<td>2</td>
</tr>
<tr>
<td>14</td>
<td>4</td>
</tr>
<tr>
<td>15</td>
<td>0</td>
</tr>
<tr>
<td>16</td>
<td>5</td>
</tr>
<tr>
<td>17</td>
<td>2</td>
</tr>
</tbody>
</table>
4 Specific Games & Questions (240 points)

4.1 Toads and Frogs (75 points)

Theorem 4.1.1. (The Death Leap Principle) Any position in which the only legal moves are jumps into isolated spaces (none of its neighbors is empty) has value zero.

Proof 4.1.1. Suppose it’s left’s turn. If she has no moves, right wins. Otherwise, she must jump a toad into a single space. Right’s response is to push the jumped-over frog forward. Now left is in the same situation as before—her only moves are jumps into single spaces. (Right may have more moves at this point, but this only makes the situation better for him.) Eventually, left has no moves, and right wins. We can argue symmetrically if right goes first.

In general note the Death Leap Principle will be used through all proofs. So one should refer to this theorem.

Problem 4.1.1. (10 points) Find with proof the game value of

\[
\begin{array}{cccccccc}
T & F & T & F & F & \square & F & T \\
\Box & \Box & T & F & F & \square & T & F & F \\
T & F & F & \square & T & T & F & F \\
T & F & F \\
T & T & T & F & F & \square & T & T
\end{array}
\]

Solution 4.1.1. This game equals

\[
F & T & \Box & \Box + \Box & T & F & F & \Box & \Box
\]

from the death leap principle. This equals

\[1 + \Box & T & F & F & \Box & \Box\]

This equals 1 from the death leap principle.

Problem 4.1.2. (20 points) Prove \((T F)^m T \square (T F)^n\) has game value \(2^{-n}\) for all \(m, n \in \mathbb{N} \cup \{0\}\).

Solution 4.1.2. We use induction on \(n\). The base case \(n = 0\) follows immediately from the Death Leap Principle. Suppose \(n > 0\). Left has only one legal move, to the position \((T F)^m \square T (T F)^n = (T F)^m \square T = 0\) by the Death Leap Principle. Similarly, right can only move to the position \((T F)^{m+1} \square T (T F)^{n-1} = 2^{-n+1}\) by the induction hypothesis.

Problem 4.1.3. (20 points) Prove for every dyadic rational \(q\), there exists a Toads and Frogs game with game value \(q\).

Solution 4.1.3. Write \(q = \frac{2k+1}{2^n}\), and WLOG \(k \geq 0\). The position 

\[(T \square (T F)^n T T F F)2k+1\]

has value \(q\) from problem 4.1.2 and the Death Leap Principle.

Problem 4.1.4. (25 points) Prove no matter whether the left or right player moves first, the game \(T^n \square F \Box F\) has game value 0 for all \(n \geq 2\).

Solution 4.1.4. The second player wins as follows.

Case 1: Left moves first. Right moves his leftmost frog on his first move, and his rightmost frog on his second move. Left’s moves are forced. Right’s second move leaves the position \(T^{n-1} F \Box T F \Box\) which has value 0 by the death leap principle. Thus, right wins moving second.

Case 2: Right moves first. Right has two options. If he moves the rightmost frog forward, left will move her rightmost toad forward three times. This forces right’s next two moves.
Left’s third move leaves the winning position $T^{n-1} F F \Box T \Box = 1$.
If right moves his leftmost frog forward, left must respond by jumping it, leaving $T^{n-1} \Box F T \Box F$. Again, right has two options. If he moves his rightmost frog, left responds by jumping it, leaving the position $T^{n-1} \Box F \Box F$, which is zero by induction. (One can check the base case $n = 2$ directly). Otherwise, right moves to the position $T^{n-2} F T F T$, and left can force the following sequence of moves:

$T^{n-2} F T F T$
$T^{n-2} F T F T$
$T^{n-2} F T F T$

The final position is 0 by the death leap principle. Thus, left wins moving second.

4.2 Partizan Splittles (90 points)

Problem 4.2.1. (5 points) Prove that if $S_L = \{1, a_1, a_2, \ldots, a_j\}$ and $S_R = \{1, b_1, b_2, \ldots, b_k\}$, where each $a_i$ and each $b_i$ is a positive odd integer, then

$$G_n = \begin{cases} 0 & \text{if } n \text{ is even} \\ * & \text{if } n \text{ is odd} \end{cases}$$

Solution 4.2.1. Each move changes the parity of the total number of tokens in all the heaps. In the final position, with zero tokens, the total is even. So when $n$ is even, there will be an even number of moves before the moves run out, so $G_n = 0$. If $n$ is odd, there will be an odd number of moves before the moves run out, so $G_n = *$.

Problem 4.2.2. (15 points) Prove that if $S_L = \{1\}$ and $S_R = \{k\}$, where $k$ is a positive odd integer, then

$$G_n = \begin{cases} n & \text{if } n < k \\ \{k - 1 \mid 0\} + G_{n-k} & \text{if } n \geq k \end{cases}$$

Solution 4.2.2. The conjectured saltus (oscillation) is $G_x$. So this theorem states that we can treat a single heap as a collection of heaps of size $k$ and a single remaining heap of size less than $k$ (possibly size 0) without changing its value. For $0 \leq a < k$, $G_a = a$ because only Left can move in such a position. And $G_k = \{k - 1 \mid 0\}$ because Right can move to the zero game, and Left can move to the game with $k - 1 < k$ tokens remaining. In a general position, it suffices to show that any move that straddles a period boundary is dominated by one that does not, because the game reduces to its “decomposed” form. Left’s moves never do so. By induction, we can show that for Right’s moves, shorter positions achieve the conjectured values and decompose at period boundaries. If $a + b = k + c$ for $0 \leq a, b, c < k$, $G_a + G_b = a + b = k + c$ but $G_a + G_b > Gk + Gc = \{k - 1 \mid 0\} + c$, so Right prefers the latter move which avoids the boundary.

Problem 4.2.3. Consider $S_L$ and $S_R$ with the properties that $1 \in S_L$ and $S_R = \{1, 3, 5, \ldots, 2k+1\}$ for some integer $k$ or $S_R = \{1, 3, 5, \ldots\}$ (i.e. all odd integers).
a) Prove that $G_n \leq G_{n+2}$ (Prop. 1)

b) Prove that $G_{2n+1} = G_{2n} + *$ (Prop. 2)

c) Prove that if $S_R$ is finite and $n - i - j \geq 2k$ is even then $G_n \leq G_i + G_j$ (Prop. 3)

Solution 4.2.3. For Prop. 1, in most sequences of play, Left wins $G_{n+2} - G_n$ by matching options naturally, removing the same number of tokens as Right does from the opposite heap. Left can then leave a position of the form $G_{n+2} - G_a - G_b$ for $a, b \geq 0$, which is $\geq 0$ by induction. The only exception is if Right removes $n + 1$ from the first heap. In this case, Left removes $n - 1$ from the second, leaving $G_1 - G_1 = 0$.

We prove Prop. 2 and Prop. 3 in tandem by induction. We assume Prop. 2 holds for $n' \leq n$ when proving Prop. 3, but that Prop. 3 holds for $n' < n$ when proving Prop. 2. For Prop. 2, we want to show the second player wins the difference game $G_{2n+1} - G_{2n} - * = G_{2n+1} - G_{2n} - G_1$. We can depict this game as:

<table>
<thead>
<tr>
<th>Left’s moves</th>
<th>Right’s moves</th>
</tr>
</thead>
<tbody>
<tr>
<td>${1, \text{others}}$</td>
<td>${1, 3, \ldots, 2k + 1}$</td>
</tr>
<tr>
<td>${1, 3, \ldots, 2k + 1}$</td>
<td>${1, \text{others}}$</td>
</tr>
</tbody>
</table>

The roles of the players are reversed in the second row, because it is the negative of the game $G_{2n} + G_1$. So in the second row, Left removes elements from $S_R$ and Right removes elements from $S_L$. If either player removes $r$ boxes from the top row, leaving a block of length $i$ odd (and possibly a second block), the other player can counter symmetrically by removing $r$ boxes from the bottom, leaving a block of length $i - 1$, which is the position 0 by induction. The reverse is true as well. The second player can respond to moves on the bottom row that leave an even-length block (refer to the picture below).

Also shown are moves $B$ which take a single box from one end of the top row, which match with the move $B'$ taking the lone box on the bottom row. So we are now left with cases that split the top row into two even-length heaps or that split the bottom row into two odd-length heaps. Only Right can do the latter because it requires removing an even number. Left’s responses parallel Right’s moves as below. If Right removes $C^R$ on the top, Left’s reply of $C^{RL}$ wins by application of Propositions 2 and 3. Similarly, Left wins after Right’s $D^R$ and Left’s $D^{RL}$ (see picture below).
We are now left with the case when Left removes an odd number from the top row, splitting it into two even-sized heaps, as in $E^L$ above. Right responds by removing as large an (odd) number as possible from one of these two even-sized heaps. If one of the heaps is of size $\leq 2k + 2$ (or $S_R$ is infinite), Right leaves that heap a singleton, canceling the single box on the second row, and wins by Prop. 3. Otherwise, he has taken away $2k + 1$ and wins by Prop. 2 and 3. Lastly, to prove Prop. 3, we show that Left wins moving second on $G_n - G_i - G_j$: 

![Diagram](image1)

The gap in the bottom row is of even length and at least $2k$. Left can respond to moves that fail to straddle the gap as below:

![Diagram](image2)

Moves outside the gap match up with moves in the other row, winning by induction. Left responds to moves inside the gap by responding on the odd side: Because the gap was of even length, and Right can remove only odd numbers, the gap is split into an even length and an odd length. Left then wins by application of Prop. 2 to both sides. A Right move that straddles the gap can only straddle one side. Left responds by removing a like number from the side below Right’s move:

![Diagram](image3)

Since the parity of the number of boxes in each row is preserved, each segment can be shortened to an even length by an even number of applications of Prop. 2, which, since $* + * = 0$ and $* = -*$, leave the game value unchanged. Left then proceeds to win by Prop. 2 applied to both sides.

**Problem 4.2.4.** (20 points) Let $n$ be odd. Prove that $H_{n+1} - H_n < 1$ and $H_n - H_{n-1} = 1$.

**Solution 4.2.4.** Induct on $n$. To prove the first statement, consider $1 + H_n - H_{n+1}$. Left can win by moving immediately to $1 + H_n$, so this value is positive. If Right moves to $1 + H_a + H_b - H_{n+1}$, Left can counter by moving to $1 + H_a + H_b - H_{n+1} - H_b$. By induction on one of the two things to prove above (depending on whether $a$ is odd or even), this is a winning move. If Right moves to $1 + H_n - H_a - H_b$, since $n + 1$ is even, $a + b$ is odd, so one of $a, b$ is odd (WLOG, say $a$). Left counters to $1 + H_{a-1} + H_b - H_a - H_b$, which is 0 by induction. This suffices for the first statement above. To prove the second statement, we show that $H_n - H_{n-1} - 1$ is a second-player win. If Right
To complete the proof of this lemma, we must show that if \((a, b)\) to any \((a_m, b_m)\) or \((b_m, a_m)\). Any such a move would necessarily have \(m < n\). But by definition of \(a_n\), we have \(a_n \neq a_m\) and \(a_n \neq b_m\). Moreover \(a_n > a_m\), so \(b_n = b_m\). Therefore \(b_n \neq a_m\) and \(b_n \neq b_m\). This leaves only the moves along the diagonal, but these are ruled out since \(b_n - a_n = n\) while \(b_m - a_m = m\).

To complete the proof of this lemma, we must show that if \((a, b)\) is not of the form \((a_n, b_n)\),
then it has a move to some \((a_n, b_n)\) or \((b_n, a_n)\). We may assume that \(a \leq b\).

First suppose \(a = a_n\) for some \(n\). If \(b > b_n\), then there is above directly to \((a_n, b_n)\).

Otherwise, \(a_n \leq b < b_n\). Let \(m = b - a\). Then \(m < n\), so \(a_m < a_n\) and there is a move to \((a_m, b_m)\).

Finally, if \(a \neq a_n\) for any \(n\), then necessarily \(a = b\). Therefore \(b \geq a > a_n\), so there is a move to \((b_n, a_n)\).

This proves the lemma.

It is easily checked that \(\phi^2 = \phi + 1\). Therefore,

\[\lfloor n\phi^2 \rfloor = \lfloor n\phi \rfloor + n\]

Moreover, we have

\[\frac{1}{\phi} + \frac{1}{\phi^2} = 1\]

so by problem 4.1.1, \(\lfloor n\phi \rfloor\) and \(\lfloor n\phi^2 \rfloor\) are complementary. This proves that

\[\lfloor n\phi \rfloor = \text{mex}\{\lfloor i\phi \rfloor, \lfloor i\phi^2 \rfloor : i < n\}\]

and

\[\lfloor n\phi^2 \rfloor = \lfloor n\phi \rfloor + n\]

and the theorem now follows from the lemma.

**Problem 4.3.3.** (15 points) Prove that every \(G\)-value \(n\) appears exactly once among all ordered pairs of piles with the first pile of any given, fixed size.

**Solution 4.3.3.** Let \(G(m, n)\) be the value of the position \((m, n)\). We notice two things:

1) \(G(m, n) \leq m + n\)
2) \(G(m, n) \geq -2n + m\).

Proof of 1):

Induction on \(a + b\). Since \(G(0, 0) = 0\) we have the basis. Assume for all \(i, j \leq m + n\), \(G(i, j) \leq i + j\). Now we want to calculate \(G(m, n)\). Let \(A\) be the set of values that need to be excluded. \(A\) contains only values < \(m + n\), so \(G(m, n) = \text{mex}(A) \leq \max(A) + 1 \leq (m + n - 1) + 1 = m + n\).

Proof of 2):

Assume that \(g = G(m, n) < m - 2n\). This means that \(g\) doesn’t appear as any \(G(k, n)\) for \(k < m\), \(m\) times in total. This can happen for three reasons: \(G(k, n) < g\), which only happens at most \(g\) times, \(g\) cannot appear because some \(G(k, j)\) for \(j < n\) equals \(g\), or \(g\) cannot appear because some \(G(k - i, g - i)\) for \(0 \leq i \leq \min(k, n)\) equals \(g\). But \(g\) appears only once in each row(by definition of mex), the second and third reasons can happen at most \(n\) times each. The total number of times \(g\) doesn’t appear is at most \(g + 2n < (m - 2n) + 2n = m\) which is a contradiction.

Now it’s easy to see that every \(g\) has to appear at least once in each row, hence it appears exactly once.

**Problem 4.3.4.** (25 points) Let \(r > 0\). \(r\)-Wythoff is played with two piles of tokens. On their turn, a player may either remove as many tokens from one pile as they wish, or remove \(a\) tokens from one pile and \(b\) from the other, where \(|a - b| < r\). Prove that the \(n^{\text{th}}\) P-position of \(r\)-Wythoff is given by \((a_n, b_n) = (\lfloor n\alpha \rfloor, \lfloor n\beta \rfloor)\) where \(\alpha = \frac{1}{2}(2 - r + \sqrt{r^2 + 4})\) and, \(\beta = \alpha + r\).
Problem 4.3.5. (250 points) Let $\phi = \frac{1 + \sqrt{5}}{2}$. Prove that

$$8 - 6\phi < a_n - \phi n < 6 - 3\phi$$

and

$$2 - 3\phi < b'_n - \phi^2 n < 6 - 3\phi$$

for $a_n \in A_1$ and $b'_n \in B'_1$.

Solution 4.3.5.

<table>
<thead>
<tr>
<th>$T_n$</th>
<th>$T_1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$D_n$</td>
<td>$A_n$</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>3</td>
<td>4</td>
</tr>
<tr>
<td>4</td>
<td>6</td>
</tr>
<tr>
<td>5</td>
<td>8</td>
</tr>
<tr>
<td>6</td>
<td>9</td>
</tr>
<tr>
<td>7</td>
<td>11</td>
</tr>
<tr>
<td>8</td>
<td>12</td>
</tr>
<tr>
<td>9</td>
<td>14</td>
</tr>
<tr>
<td>10</td>
<td>16</td>
</tr>
<tr>
<td>11</td>
<td>17</td>
</tr>
</tbody>
</table>

Figure 1: Table 1
The above tables represent a brute force of the beginning of $T_1$ and $T_0$ and feasible integrals.

We first prove a bunch of lemmas.

**Lemma 1:** For all $n \geq 2$, $x \geq a_n$ iff $b'_{x-n+1} \geq x$.

**Proof:** Suppose $x \geq a_n$. Since $B'$ strictly increasing,

$$b_{x-n+1} \geq b'_{a_n-n+1} + (x - a_n) > a_n + (x - a_n) = x$$

Now suppose $x < a_n$. Then

$$b_{x-n+1} \leq b'_{a_n-n} - (a_n - x - 1) < a_n - (a_n - x - 1) = x + 1$$

Proving lemma 1.

**Lemma 2:** For all $n \geq 2$, $y \leq a_n$ iff $b'_{y-n} < y$.

**Proof:** Let $y \leq a_n$. Then

$$b'_{y-n} \leq b_{a_n-n} - (a_n - y) = y$$

Now let $y > a_n$. Then

$$b'_{y-n} \geq b'_{a_n-n+1} + (y - a_n - 1) > y - 1$$
Proving lemma 2.

Lemma 3: Suppose $(a_m, b_m)$ is the initial pair of an integral $I$. Then $a_m = |m\phi|$ iff $|I| > 1$.
Proof: The hypothesis implies $mex\{b_i - a_i : i < m\} = m$. Note that $a_m + m > a_{m-i} + (m-1) \geq b_{m-i}$ for all $i \geq 1$. Assume $|I| = 1$. Then $p = (a_m, a_m + m) \in T_1$. If $a_m = |m\phi|$, then $p \in T_0$, a contradiction. Now assume $|I| > 1$. If $a_m \neq |m\phi|$, then $p$ would be an integral of size 1 in $T_1$, since then $p \notin T_0$. This proves lemma 3.

Lemma 4: Suppose $(a_m, b_m)$ is the initial pair of an integral $I$. Then $b'_m = b_m$.
Proof: We already remarked that $i < m$ implies $b_i < b_m$. Table 2 shows that $b_m$ is the smallest $B_1$-member for every integral $I$. Hence also $b_j > b_m$ for every $j > m$. It follows that $b'_m = b_m$. This proves lemma 4.

Both assertions hold for $n \leq 5$ as can be seen from Table 1. We assume the two statements to hold for all $i < n$ and $n \geq 6$. For the first inequality we find suitable $x$ and $y$ satisfying $b'_{x-n+1} > x, b'_{y-n} < y$ and then use Lemma 1 and 2 to conclude $y \leq a_n \leq x$. For proving the second inequality we lean on Theorem 4.1.1 and also use the first inequality.

We begin with $a_n$. Let $x = n\phi + s$, where $s = [(n-3)\phi] + 5 - n\phi$. Since $n(\phi - 1) + s + 1 < n$ for all $n \geq 6$, we can apply the induction hypothesis on the second to write

$$b'_{x-n+1} = b_{n(\phi-1)+s+1} > (n\phi - 1) + s + 1)(\phi + 1) + 2 - 3\phi \geq x$$

as a straightforward computation shows. Therefore $x \geq a_n$, i.e.,

$$a_n \leq n\phi + s < n\phi + (n-3)\phi + 6 - n\phi = n\phi + 6 - 3\phi$$

as required. Now let $y = n\phi + t$, where $t = [(n-6)\phi] + 9 - n\phi$. Since $n(\phi - 1) + t < n$ for all $n \geq 0$ we can again use induction to get

$$b'_{y-n} = b'_{n(\phi-1)+t} < (n(\phi - 1) + t)(\phi + 1) + 6 - 3\phi \leq y$$

Hence $y \leq a_n$, i.e.,

$$a_n \geq n\phi + t > n\phi + (n-6)\phi + 8 - n\phi = n\phi + 8 - 6\phi$$

as required.

For proving the second inequality, it is convenient to have the following lemma.

Lemma 5: If $b'_n - a_n \leq n$ then the upper bound of the second inequality holds; and if $b'_n - a_n \geq n - 1$, then the lower bound of the second inequality holds.
Proof: To prove this lemma, using the first ineq. write

$$b'_n \leq a_n + n < n(\phi + 1) + 6 - 3\phi$$

$$b'_n \geq a_n + n - 1 > n(\phi + 1) + 7 - 6\phi > n(\phi + 1) + 2 - 3\phi$$

proving the lemma.

If $(a_n, b_n)$ constitutes an integral of size 1, then $b'_n = b_n$ by lemma 4; hence $b_n - a_n = n = b'_n - a_n$; thus both bounds hold by lemma 5. So assume henceforth that $(a_n, b_n)$ is an integral of size $> 1$.

Throughout the rest of the proof, $(a_{d+1}, b_{d+1})$ will denote the initial pair of the integral $I$. 

...
containing \((a_n, b_n)\). We consider several cases.

Case 1: \(n = d + 1\). Table 2 shows that then \(b_n - a_n = n + 1\). Hence by lemma 4,
\[
b'_n = \lceil n\phi \rceil + n + 1 < n(\phi + 1) + 1 < n(\phi + 1) + 6 - 3\phi
\]
\[
b'_n > n(\phi + 1) > n(\phi + 1) + 2 - 3\phi
\]

Case 2: \(n = d + 2\). If \(|I| = 2\), then table 3 shows that again \(b'_n = b_n\) and \(b_n - a_n = n - 1\), hence the assertion follows from lemma 5. So assume \(|I| > 2\). From table 3,
\[
b'_{d+1} = b_{d+1} = a_{d+1} + d + 2 < a_{d+1} + d + 4 = b_{d+2}
\]
Hence
\[
b'_n = b'_{d+2} \leq b_{d+2} = \lceil (d+1)\phi \rceil + d + 4 < n(\phi + 1) + 2 - \phi < n(\phi + 1) + 6 - 3\phi
\]

On the other hand,
\[
b'_n > b'_{n-1} = b'_{d+1} = a_{d+1} + d + 2
\]
do
\[
b'_n \geq a_{d+1} + d + 3 > (d + 1)\phi + d + 2 = n(\phi + 1) - \phi > n(\phi + 1) + 2 - 3\phi
\]

Case 3: \(n = d + 3\). If \(|I| = 3\) then table 2 shows that
\[
b'_n = \max(a_{d+1} + d + 4, a_{d+3} + d + 1) \leq a_{d+3} + d + 2 = a_n + n - 1
\]
The upper bound now follows from lemma 5. So assume \(|I| > 3\). Then
\[
b'_n = b'_{d+3} \leq b_{d+3} = a_{d+1} + d + 7 < (d + 1)\phi + d + 7 = n(\phi + 1) + 4 - 2\phi < n(\phi + 1) + 6 - 3\phi
\]
For computing a lower bound, note that Table 2 implies
\[
b'_{n-2} \geq 2 = b_{d+1} + 2 = b'_{d+1} + 2 = a_{d+1} + d + 4
\]
If this ineq. is strict, then
\[
b'_n \geq a_{d+1} + d + 5 > (d + 1)\phi + d + 4 = n(\phi + 1) + 1 - 2\phi > n(\phi + 1) + 2 - 3\phi
\]
Otherwise, \(b'_{d+3} = a_{d+1} + d + 4\). From table 2, \(a_{d+4} \geq a_{d+1} + 4\), so \(b_{d+4} \geq a_{d+1} + d + 5\). We further see from table 3 that
\[
b_{d+5} \geq a_{d+1} + d + 8
\]
\[
b_{d+6} \geq a_{d+1} + d + 13
\]
We will refer to these as (3). Hence, if \(|I| > 3\), then \(b'_{d+2} = a_{d+1} + d + 4 < b'_{d+3}\). Therefore we must have \(|I| = 3\). Then \(b'_{d+2} = a_{d+3} + d + 1 = a_{d+1} + d + 3\), hence \(b'_{d+3} = b'_{d+2} + 1 = a_{d+3} + d + 2\). Equivalently, \(b'_n = a_n + n - 1\), so the lower bound for this subcase follows from lemma 5.

Case 4: \(n = d + 4\). Table 3 and (3) show that then \(b'_{d+4} \geq a_{d+1} + d + 7\). Thus
\[
b'_n > (d + 1)\phi + d + 6 = n(\phi + 1) + 2 - 3\phi
\]
For computing an upper bound, we use the first inequality to get

\[ a_{d+4} < (d + 4)\phi + 6 - 3\phi = (d + 1)\phi + 6 < \lfloor (d + 1)\phi \rfloor + 7 \]

Thus by lemma 3 \( a_{d+4} \leq a_{d+1} + 6 \). If \(|I| = 4\) we then have \( b_{d+4} \leq a_{d+1} + d + 7 \), so \( b'_{d+4} = a_{d+1} + d + 7 \).

We now assume \(|I| > 4\). If \( a_{d+5} > a_{d+1} + 9 \), then \( \{a_{d+1} + i : i = 2, 4, 5, 7, 8, 9\} \subseteq B_1 \), which leads to a contradiction as at the end of the proof of theorem 4.1.I. Therefore \( a_{d+5} \leq a_{d+1} + 9 \), so \( b_{d+5} \leq a_{d+1} + d + 10 \). By (3),

\[ a_{d+1} + d + 8 \leq b_{d+5} \leq a_{d+1} + d + 10 \]

Which we refer to as (4).

It follows that \( b'_{d+4} = b_{d+5} \leq a_{d+1} + d + 10 \), so \( b'_{d+4} \leq a_{d+1} + d + 10 \) golds for \(|I| \geq 4\). Hence

\[ b'_n = b'_{d+4} < (d + 1)\phi + d + 10 = n(\phi + 1) + 6 - 3\phi \]

Case 5: \( n = d + 5 \). By (3) and (4) we have \( b'_{d+5} = a_{d+1} + d + 11 \) if \(|I| = 5\), and \( b'_{d+5} = a_{d+1} + d + 12 \) if \(|I| = 6\). Hence

\[ b'_n \leq a_{d+1} + d + 12 = a_{d+4} + d + 6 \leq a_{d+5} + d + 5 = a_n + n \]

and the upper bound follows from lemma 5. For getting a lower bound we write

\[ b'_n = b'_{d+5} \geq a_{d+1} + d + 11 > (d + 1)\phi + d + 10 = n(\phi + 1) + 5 - 4\phi > n(\phi + 1) + 2 - 3\phi \]

Case 6: \( n = d + 6 \). From table 2, \( b'_n = a_{d+6} + d + 5 = a_n + (n - 1) \) so both bounds follow from lemma 5.