1. The product of the positive factors of a positive integer \( n \) is 8000. What is \( n \)?

*Proposed by: Jacob Wachspress, Nathan Bergman*

**Answer:** \( 20 \)

20 has 6 factors, 3 pairs, this is \( 20^3 = 8000 \).

2. The least common multiple of two positive integers \( a \) and \( b \) is \( 2^5 \times 3^5 \). How many such ordered pairs \((a, b)\) are there?

*Proposed by: Rahul Saha*

**Answer:** \( 121 \)

Looking at each prime, there are 11 choices, so the answer is \( 11^2 \).

3. Let \( f \) be a function over the natural numbers so that

1. \( f(1) = 1 \)

2. If \( n = p_1^{e_1} \cdots p_k^{e_k} \) where \( p_1, \cdots, p_k \) are distinct primes, and \( e_1, \cdots e_k \) are non-negative integers, then \( f(n) = (-1)^{e_1 + \cdots + e_k} \).

Find \( \sum_{i=1}^{2019} \sum_{d|i} f(d) \).

*Proposed by: Marko Medvedev*

**Answer:** \( 44 \)

Since the function is completely multiplicative, \( \sum_{d|i} f(d) \) is given by product of \( \frac{f(p_k)^{x_k+1} - 1}{f(p_k) - 1} \)

which is 0 if \( x_k \) is odd and 1 if \( x_k \) is even (recall that \( f(p) = -1 \) for all primes \( p \)). Therefore the required sum evaluates to the number of squares less than 2019, which is 44.

4. Let \( n \) be the smallest positive integer which can be expressed as a sum of multiple (at least two) consecutive integers in precisely 2019 ways. Then \( n \) is the product of \( k \) not necessarily distinct primes. Find \( k \).

*Proposed by: Oliver Thakar*

**Answer:** \( 105 \)

\( n \) can be written as a sum of \( 2k + 1 \) consecutive integers if and only if \( 2k + 1 \) is a divisor of \( n \), for letting \( x \) be the integer in the center of the sum, then \( n = (x-k) + \cdots + x + \cdots + (x+k) = (2k+1)x \). Hence, the number of odd divisors of \( n \) minus one (1 is an odd divisor of \( n \) but does not correspond to a sum of at least two consecutive integers) equals the number of ways that \( n \) can be written as a sum of an odd number of consecutive integers.

There is a bijection between each writing of \( n \) as a sum of consecutive even integers and the odd divisors of \( n \).

Letting \( 2k \) be an even integer, and \( x \) being the center of the sum (so that \( x \) is some integer plus \( \frac{1}{2} \)) then:

\[
    n = (x - \frac{1}{2} - k) + \cdots + (x - \frac{1}{2}) + (x + \frac{1}{2}) + \cdots + (x + \frac{1}{2} + k) = kx.
\]

Thus, we know that the factor of 2 in \( k \) must be one more than the factor of 2 in \( n \), and that \( 2k \) divides \( n \). For each odd divisor of \( n \), there is one such \( k \) which is a power of 2 times that divisor, and vice versa.
5. Consider the first set of 38 consecutive positive integers who all have sum of their digits not divisible by 11. Find the smallest integer in this set.

Proposed by: Marko Medvedev

Answer: 999981

Consider first the last two digits. Note that if we don’t go past a multiple of 100, then we will have a string of at least 12 consecutive sums of digits since we will have a number ending in zero such that 29 plus that number has sum of digits 11 more than that number. Note that if we go up to at least 19 mod 100 then we will have 11 consecutive sums, and if we go down to at most 80 then we will have 11 consecutive sums, so we must have the range from 100x + 81 to 100x + 118. Then we must have the sum of digits of 100x + 100 must have sum 1 mod 11, so x + 1 has sum of digits 1 mod 11, and 100x + 81 must have sum 1 mod 11 so x has sum 3 mod 11. Thus when we add 1 to x we have to increase digitsum by 9 mod 11. Note that x must end in some number of nines. If it ends in k nines, then we increase by 1 − 9k Thus 2k + 1 = 9 (mod 11) so k = 4 so the smallest x is 9999 and our answer is 999981.

6. Let f be a polynomial with integer coefficients of degree 2019 such that the following conditions are satisfied:

1. For all integers n, f(n) + f(−n) = 2.
2. 101^2 | f(0) + f(1) + f(2) + ⋯ + f(100). Compute the remainder when f(101) is divided by 101^2.

Proposed by: Matthew Kendall

Answer: 203

We use this fact: For nonnegative integer k and prime p > 2,

\[ p^2 | 1^{2k+1} + 2^{2k+1} + ⋯ + (p-1)^{2k+1}. \]

This comes from \( j^{2k+1} + (p-j)^{2k+1} \equiv (2k+1)jp \pmod{p^2} \) and summing over all j.

Let \( p = 101 \). Plugging n = 0 into 1 gives f(0) = 1. Since deg f = 2019, we can write f(n) = 1 + an + g(n) where g is odd and all of its terms are of degree at least 3. Now using fact 2, \( p^2 | g(0) + g(1) + g(2) + ⋯ + g(p) \). This means

\[ f(0) + f(1) + f(2) + ⋯ + f(100) \equiv p + a(1 + ⋯ + (p-1)). \pmod{p^2} \]

So 2p + ap(p − 1) ≡ 0 \pmod{p^2} or ap ≡ 2p \pmod{p^2}. Hence, f(p) = 1 + ap ≡ 2p + 1 \pmod{p^2}.

Plugging in p = 101 gives f(101) ≡ 203 \pmod{101^2}. 
7. For a positive integer \( n \), let \( f(n) = \sum_{i=1}^{n} \left\lfloor \log_2 n \right\rfloor \). Find the largest \( n < 2018 \) such that \( n \mid f(n) \).

**Proposed by: Eric Neyman**

**Answer:** 1013

First note that

\[
f(2^{r+1} - 1) = \sum_{k=0}^{r} k \cdot 2^k = \sum_{i=1}^{r} \sum_{j=1}^{2^i} (2^{r+1} - 2^i) = (r - 1)2^{r+1} + 2.
\]

Thus, if we write \( n = 2^{r+1} - 1 + m \), where \( 0 \leq m \leq 2^{r+1} \), we have

\[
f(n) = (r - 1)2^{r+1} + 2 + m(r + 1).
\]

Thus, the condition \( n \mid f(n) \) is equivalent (after subtracting \( (r - 1)n \) from \( f(n) \)) to

\[
2^{r+1} - 1 + m \mid 2 + m(r + 1) + r - 1 - m(r - 1) = 2m + r + 1.
\]

Now, the right-hand side is more than zero times the left-hand side but more than twice the left-hand side, so \( n \mid f(n) \) if and only if \( 2^{r+1} - 1 + m = 2m + r + 1 \), i.e. \( m = 2^{r+1} - r - 2 \), so \( n = 2^{r+2} - r - 3 \).

The largest such value that is less than 2018 is \( 2^{10} - 8 - 3 = 1013 \).

8. Call a positive integer \( n \) **compact** if for any infinite sequence of distinct primes \( p_1, p_2, \ldots \) there exists a finite subsequence of \( n \) primes \( p_{x_1}, p_{x_2}, \ldots, p_{x_n} \) (where the \( x_i \) are distinct) such that

\[
p_{x_1}p_{x_2} \cdots p_{x_n} \equiv 1 \pmod{2019}
\]

Find the sum of all **compact** numbers less than \( 2 \cdot 2019 \).

**Proposed by: Rahul Saha**

**Answer:** 14112

**Claim 1:** Let \( n \) be a compact number. Then we must have \( a^n \equiv 1 \pmod{2019} \) for all \((a, 2019) = 1\).

**Proof:** By Dirichlet’s theorem on arithmetic progressions, we can find infinitely many primes \( p \equiv a \pmod{2019} \). Letting our sequence be composed only of these primes, we must have \( a^n \equiv 1 \pmod{2019} \).

**Claim 2:** If \( a^n \equiv 1 \pmod{2019} \) for all \((a, 2019) = 1\), then \( n \) is a compact number.

**Proof:** Note that by taking all large enough primes in our sequence, we can assume \((p_i, 2019) = 1\). But some residue \( a \pmod{2019} \) must appear infinitely many times, which gives us \( a^n \equiv 1 \pmod{2019} \), as desired.

**Claim 3:** Let \( n \) be the minimal compact number. Then all compact numbers are multiples of \( n \), and conversely any multiple of \( n \) is a good number.

**Proof:** Let \( N \) be another compact number, and suppose \( N = nq + r \), but then we have \( a^N \equiv a^r \equiv 1 \) which would make \( r \) the minimal good number, a contradiction unless \( r = 0 \).

The other direction is trivial.

**Claim 4:** The minimal compact number is 672.

**Proof:** Let \( x \) and \( y \) be primitive roots modulo 3 and 673. Then the order of \( xy \) is \( \frac{2 \cdot 672}{(2, 672)} = 672 \), so the minimal compact number is at least 672. Note, \( a^{672} \equiv 1 \pmod{3} \) and \( a^{672} \equiv 1 \pmod{673} \) therefore \( a^{672} \equiv 1 \pmod{2019} \) for all \((a, 2019) = 1\). Therefore the minimal compact number is 672.

Therefore, the sum is \( 672 \cdot (1 + 2 + 3 + 4 + 5 + 6) = 672 \cdot 21 = 14112 \).