Algebra B

1. Let \(a, b\) be positive integers such that \(a + b = 10\). Let \(\frac{p}{q}\) be the difference between the maximum and minimum possible values of \(\frac{1}{a} + \frac{1}{b}\), where \(p\) and \(q\) are relatively prime positive integers. Compute \(p + q\).

2. If \(x\) is a real number so \(3^x = 27x\), compute \(\log_3\left(\frac{3^x}{x^2}\right)\).

3. Let \(x\) and \(y\) be positive real numbers that satisfy \((\log x)^2 + (\log y)^2 = \log(x^2) + \log(y^2)\). Compute the maximum possible value of \((\log xy)^2\).

4. Let \(f(x) = x^2 + 4x + 2\). Let \(r\) be the difference between the largest and smallest real solutions of the equation \(f(f(f(x)))) = 0\). Then \(r = a^p\) for some positive integers \(a, p, q\) so \(a\) is square-free and \(p, q\) are relatively prime positive integers. Compute \(a + p + q\).

5. Let \(Q\) be a quadratic polynomial. If the sum of the roots of \(Q^{100}(x)\) (where \(Q^i(x)\) is defined by \(Q^1(x) = Q(x), Q^i(x) = Q(Q^{i-1}(x))\) for integers \(i \geq 2\)) is 8 and the sum of the roots of \(Q\) is \(S\), compute \(|\log_2(S)|\).

6. Let \(\mathbb{N}_0\) be the set of non-negative integers. There is a triple \((f, a, b)\), where \(f\) is a function from \(\mathbb{N}_0\) to \(\mathbb{N}_0\) and \(a, b \in \mathbb{N}_0\), that satisfies the following conditions:
   1) \(f(1) = 2\)
   2) \(f(a) + f(b) \leq 2\sqrt{f(a)}\)
   3) For all \(n > 0\), we have \(f(n) = f(n-1)f(b) + 2n - f(b)\)

   Find the sum of all possible values of \(f(b + 100)\).

7. Let \(\omega = e^{\frac{2\pi i}{2017}}\) and \(\zeta = e^{\frac{2\pi i}{2019}}\). Let \(S = \{(a, b) \in \mathbb{Z} \mid 0 \leq a \leq 2016, 0 \leq b \leq 2018, (a, b) \neq (0, 0)\}\). Compute \(\prod_{(a, b) \in S} (\omega^a - \zeta^b)\).

8. A weak binary representation of a nonnegative integer \(n\) is a representation \(n = a_0 + 2 \cdot a_1 + 2^2 \cdot a_2 + \ldots\) such that \(a_i \in \{0, 1, 2, 3, 4, 5\}\). Determine the number of such representations for 513.