Team Round

This Team Round consists of 15 questions. The maximum possible score on the round is 90 points. Your team will have **35** minutes to read these instructions and attempt the problems. Your response to each problem will consist of two nonnegative integers. The first integer will be your team's answer to the question. The second integer must be a member of the set $\{4, 5, 6\}$, and this is the number of points your team will receive for a correct answer to the problem. If you elect for a problem to be graded out of 4 points and do not give a correct answer, your team will receive 0 points for that problem. If you elect for a problem to be graded out of 5 points and do not give a correct answer, your team will receive -2 points for that problem. If you elect for a problem to be graded out of 6 points and do not give a correct answer, your team will receive -8 points for that problem. If your team scores fewer than 0 points on the Team Round, we will graciously round your score up to 0. (However, we do not think this will be a winning score.) Good luck!

1. Two unit squares are stacked on top of one another to form a 1×2 rectangle. Each of the seven edges is colored either red or blue. How many ways are there to color the edges in this way such that there is exactly one path along all-blue edges from the bottom-left corner to the top-right corner?

2. In a standard game of Rock–Paper–Scissors, two players repeatedly choose between rock, paper, and scissors, until they choose different options. Rock beats scissors, scissors beats paper, and paper beats rock. Nathan knows that on each turn, Richard randomly chooses paper with probability 33%, scissors with probability 44%, and rock with probability 23%. If Nathan plays optimally against Richard, the probability that Nathan wins is expressible as a/b where a and b are coprime positive integers. Find $a + b$.

3. Julia is placing identical 1-by-1 tiles on the 2-by-2 grid pictured, one piece at a time, so that every piece she places after the first is adjacent to, but not on top of, some piece she's already placed. Determine the number of ways that Julia can complete the grid.

4	3
1	2

4. What is the sum of the leading (first) digits of the integers from 1 to 2019 when the integers are written in base 3? Give your answer in base 10.

5. Let $f(x) = x^3 + 3x^2 + 1$. There is a unique line of the form $y = mx + b$ such that $m > 0$ and this line intersects $f(x)$ at three points, A, B, C such that $AB = BC = 2$. Find $\lfloor 100m \rfloor$.

6. Pavel and Sara roll two, fair six-sided dice (with faces labeled from 1 to 6) but do not look at the result. A third-party observer whispers the product of the face-up numbers to Pavel and the sum of the face-up numbers to Sara.

   Pavel and Sara are perfectly rational and truth-telling, and they both know this.

   Pavel says, "With the information I have, I am unable to deduce the sum of the two numbers rolled."

   Sara responds, "Interesting! With the information I have, I am unable to deduce the product of the two numbers rolled."

   Pavel responds, "Wow! I still cannot deduce the sum. But I'm sure you know the product by now!"

What is the product?

7. For all sets A of complex numbers, let $P(A)$ be the product of the elements of A. Let $S_z = \left\{1, 2, 9, 99, 999, \frac{1}{z}, \frac{1}{z^2}\right\}$, let T_z be the set of nonempty subsets of S_z (including S_z), and let $f(z) = 1 + \sum_{s \in T_z} P(s)$. Suppose $f(z) = 6125000$ for some complex number z. Compute the product of all possible values of z.

8. The curves $y = x + 5$ and $y = x^2 - 3x$ intersect at points A and B. C is a point on the lower curve between A and B. The maximum possible area of the quadrilateral $ABCO$ can be written as $\frac{A}{B}$ for coprime A, B. Find $A + B$.

9. Find the integer $\sqrt[5]{55^5 + 3183^5 + 28969^5 + 85282^5}$.

10. Define the unit N-hypercube to be the set of points $[0,1]^N \subset \mathbb{R}^N$. For example, the unit 0-hypercube is a point, and the unit 3-hypercube is the unit cube. Define a k-face of the unit N-hypercube to be a copy of the k-hypercube in the exterior of the N-hypercube. More formally, a k-face of the unit N-hypercube is a set of the form

$$\prod_{i=1}^{N} S_i$$

where S_i is either $\{0\}$, $\{1\}$, or $[0,1]$ for each $1 \le i \le N$, and there are exactly k indices i such that $S_i = [0,1]$.

The expected value of the dimension of a random face of the unit 8-hypercube (where the dimension of a face can be any value between 0 and N) can be written in the form $\frac{p}{q}$ where p and q are relatively prime positive integers. Find $p + q$.

11. The game Prongle is played with a special deck of cards: on each card is a nonempty set of distinct colors. No two cards in the deck contain the exact same set of colors. In this game, a "Prongle" is a set of at least 2 cards such that each color is on an even number of cards in the set. Let k be the maximum possible number of prongles in a set of 2019 cards. Find $\lfloor \log_2(k) \rfloor$.

12. In quadrilateral $ABCD$, angles A, B, C, D form an increasing arithmetic sequence. Also, $\angle ACB = 90°$. If $CD = 14$ and the length of the altitude from C to AB is 9, compute the area of $ABCD$.

13. Let $e_1, e_2, \ldots e_{2019}$ be independently chosen from the set $\{0, 1, \ldots, 20\}$ uniformly at random. Let $\omega = e^{\frac{2\pi}{i} 2019}$. Determine the expected value of $|e_1\omega + e_2\omega^2 + \ldots + e_{2019}\omega^{2019}|^2$.

14. Consider a grid of black and white squares with 3 rows and n columns. If there is a non-empty sequence of white squares $s_1, \ldots, s_m$ such that s_1 is in the top row and s_m is in the bottom row and consecutive squares in the sequence share an edge, then we say that the grid percolates. Let T_n be the number of grids which do not percolate. There exists constants a, b such that $\frac{T_n}{ab^n}$ approaches 1 as n approaches ∞. Then b is expressible as $(x + \sqrt{y})/z$ for squarefree y and coprime x, z. Find $x + y + z$.

15. Determine the number of functions $f : \mathbb{Z}^+ \to \mathbb{Z}^+$ so that for all positive integers x we have $f(f(x)) = f(x+1)$, and $\max(f(2), \ldots, f(14)) \le f(1) - 2 = 12$.