1. Let \( f(x) = \frac{x^2 + a}{x + b} \) satisfy \( f(f(f(x))) = x \) for real numbers \( a, b \). If the maximum value of \( a \) is \( \frac{p}{q} \), where \( p, q \) are relatively prime integers, what is \( |p| + |q| \)?

2. Let \( C \) denote the curve \( y^2 = \frac{x(x+1)(2x+1)}{6} \). The points \( (\frac{1}{2}, a), (b, c), \) and \( (24, d) \) lie on \( C \) and are collinear, and \( ad < 0 \). Given that \( b, c \) are rational numbers, find \( 100b^2 + c^2 \).

3. Let \( \{x\} = x - \lfloor x \rfloor \). Consider a function \( f \) from the set \( \{1, 2, \ldots, 2020\} \) to the half-open interval \( [0, 1) \). Suppose that for all \( x, y \), there exists a \( z \) so that \( \{f(x) + f(y)\} = f(z) \). We say that a pair of integers \( m, n \) is valid if \( 1 \leq m, n \leq 2020 \) and there exists a function \( f \) satisfying the above so \( f(1) = \frac{m}{n} \). Determine the sum over all valid pairs \( m, n \) of \( mn \).

4. Let \( P \) be a 10-degree monic polynomial with roots \( r_1, r_2, \ldots, r_{10} \neq 0 \) and let \( Q \) be a 45-degree monic polynomial with roots \( \frac{1}{r_i} + \frac{1}{r_j} - \frac{1}{r_ir_j} \) where \( i < j \) and \( i, j \in \{1, \ldots, 10\} \). If \( P(0) = Q(1) = 2 \), then \( \log_2(|P(1)|) \) can be written as \( \frac{a}{b} \) for relatively prime integers \( a, b \). Find \( a + b \).

5. Suppose we have a sequence \( a_1, a_2, \ldots \) of positive real numbers so that for each positive integer \( n \), we have that \( \sum_{k=1}^{n} a_k \lfloor \sqrt{k} \rfloor = n^2 \). Determine the first value of \( k \) so \( a_k > 100 \).

6. Given integer \( n \), let \( W_n \) be the set of complex numbers of the form \( re^{2q\pi i} \), where \( q \) is a rational number so that \( qn \in \mathbb{Z} \) and \( r \) is a real number. Suppose that \( p \) is a polynomial of degree \( \geq 2 \) such that there exists a non-constant function \( f : W_n \to \mathbb{C} \) so that \( p(f(x)p(f(y)) = f(xy) \) for all \( x, y \in W_n \). If \( p \) is the unique monic polynomial of lowest degree for which such an \( f \) exists for \( n = 65 \), find \( p(10) \).

7. Suppose that \( p \) is the unique monic polynomial of minimal degree such that its coefficients are rational numbers and one of its roots is \( \sin \frac{2\pi}{7} + \cos \frac{4\pi}{7} \). If \( p(1) = \frac{a}{b} \), where \( a, b \) are relatively prime integers, find \( |a + b| \).

8. Let \( a_n \) be the number of unordered sets of three distinct bijections \( f, g, h : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\} \) such that the composition of any two of the bijections equals the third. What is the largest value in the sequence \( a_1, a_2, \ldots \) which is less than 2021?