Number Theory A Solutions

1. Compute the last two digits of $9^{2020} + 9^{2020}^2 + \ldots + 9^{2020}^{2020}$.

   \textit{Proposed by: Nancy Xu}

   \textbf{Answer: 20}

   It is enough to compute the residue of $9^{2020} + 9^{2020}^2 + \ldots + 9^{2020}^{2020}$ modulo 100. We have:

   \begin{align*}
   9^{2020} & \equiv (10 - 1)^{2020} \pmod{100} \\
   & \equiv \sum_{n=0}^{2020} \binom{2020}{n} (10)^n (-1)^{2020-n} \pmod{100} \\
   & \equiv \binom{2020}{1} (10)(-1)^{2019} + (-1)^{2020} \pmod{100} \\
   & \equiv -20200 + 1 \pmod{100} \\
   & \equiv 1 \pmod{100}.
   \end{align*}

   Then $9^{2020k} \equiv 1^k \pmod{100} \equiv 1 \pmod{100}$ for all $k$, so $9^{2020} + 9^{2020}^2 + \ldots + 9^{2020}^{2020} \equiv 2020 \equiv 20 \pmod{100}$.

2. How many ordered triples of nonzero integers $(a, b, c)$ satisfy $2^{abc} = a + b + c + 4$?

   \textit{Proposed by: Austen Mazenko}

   \textbf{Answer: 6}

   Since $2ab - 1 \neq 0$ for integers $a, b$, we need $c = \frac{a+b+4}{2ab-1}$ to be an integer. If $|a|, |b| \geq 2$ then $|2ab - 1| > |a + b + 4|$ unless $a = b = 2$, so $c = 5$. Thus, one of $a, b$ is in $\{-1, 1\}$. If $a = 1$, then $(2b - 1)((b + 5)$ and $b = 1, 6$, giving $(1, 1, 6)$ and cyclic permutations. If $a = -1$, then $(2b + 1)(b + 3)$, so $b = -1$ or $b = 2$. In either case, we get $(-1, -1, 2)$ and cyclic permutations. This exhausts all possible cases, so our answer is 6.

3. Find the sum (in base 10) of the three greatest numbers less than 1000_{10} that are palindromes in both base 10 and base 5.

   \textit{Proposed by: Henry Erdman}

   \textbf{Answer: 1584}

   Noting that $2 \times 5^4 > 1000$, first we consider palindromes of the form $1XXX1_5$. Such numbers are greater than $5^4 = 625$. Note, however, that the final digit (in base 10) must be congruent to 1 modulo 5, so the greatest palindrome in both bases is of the form $6X6_{10}$. Thus we have ten options, and by trial and error, we find $676_{10} = 102015$ and $626_{10} = 100015$. These are the two largest numbers that satisfy our conditions, so we only have to find the next-largest. Note that any number greater than 4000_5 is also greater than 500_{10} and thus cannot be a palindrome in base 10 as well, since we have no number $500_{10} < x < 625_{10}$ such that the first and last digits match and are congruent to 4 modulo 5. Similarly, for $x > 3000_5$, we need $375_{10} < x < 500_{10}$ and the first and last digits of $x$ to be congruent to 3 modulo 5. The only such palindromes are 383_{10} and 393_{10}, neither of which are palindromes in base 5. Moving down to the range $2000_5 = 250_{10} < x < 375_{10}$, $292_{10} = 2132_5$ is not a palindrome in base 5, but $282_{10} = 2112_5$ is, thus we have found our third number. Summing in base 10, $676 + 626 + 282 = 1584$.

4. Given two positive integers $a \neq b$, let $f(a, b)$ be the smallest integer that divides exactly one of $a, b$, but not both. Determine the number of pairs of positive integers $(x, y)$, where $x \neq y$, $1 \leq x, y \leq 100$ and $\gcd(f(x, y), \gcd(x, y)) = 2$. 

1
5. We say that a positive integer $n$ is divisible if there exist positive integers $1 < a < b < n$ such that, if the base-$a$ representation of $n$ is $\sum_{i=0}^{k_1} a_i a^i$, and the base-$b$ representation of $n$ is $\sum_{i=0}^{k_2} b_i b^i$, then for all positive integers $c > b$, we have that $\sum_{i=0}^{k_2} b_i c^i$ divides $\sum_{i=0}^{k_1} a_i c^i$. Find the number of non-divisible $n$ such that $1 \leq n \leq 100$.

Proposed by: Frank Lu

Answer: 27

First, note that if $n$ can be written as $pq$, where $1 < p < q$ are positive integers, then note that the base $n - 1$ representation of $n$ is $\sum_{i=0}^{k_1} a_i a^i$, and the base $q - 1$ representation of $n$ is $\sum_{i=0}^{k_2} b_i b^i$, then for all positive integers $c > b$, we have that $\sum_{i=0}^{k_2} b_i c^i$ divides $\sum_{i=0}^{k_1} a_i c^i$. Thus, we only need to consider the positive integers that aren’t primes or square of primes.

Also, for $p > 2$, we see that base $p - 1$ yields that $p^2$ gives $(p - 1)^2 + 2(p - 1) + 1$, and base $p^2 - 1$ yields $p^2 - 1 + 1$, so thus for $c \geq p^2 - 1$ we have that $(c + 1)|(c^2 + 2c + 1)$.

Now, given integer $n$ and base-$a$, suppose that the base-$a$ representation of $n$ is $\sum_{i=0}^{k} a_i a^i$, let $p_{a,n}(x)$ be the polynomial $\sum_{i=0}^{k} a_i x^i$. Then, note that if we write $p_{a,n}(x)$ as $p_{b,n}(x)q(x) + r(x)$, where $r(x)$ has degree less than $p_{b,n}(x)$. But then note that for sufficiently large $x$, $p_{b,n}(x) > r(x)$.

But then, we see that if $r(x) \neq 0$, then we see that for each integer $x > n$ that $p_{b,n}(x)r(x)$ implies that $r(x) = 0$ for all $x$ sufficiently large. But then $r$ is the zero polynomial, giving that $p_{b,n}(x)|p_{a,n}(x)$.

If $p_{b,n}(x)$ and $p_{a,n}(x)$ are the same degree, we see that the latter is a scalar multiple of the former, by say, $c$. But then we see that $c < p$ and we need $c|p$, contradiction.

Otherwise, note then that if the degree of $p_{a,n}(x)$ is $d$, then note then that $1 < p_{b,n}(a) < a^d \leq p_{a,n}(a) = n$, which means that $n$ isn’t prime, contradiction.

Thus, we see that the only non-divisible numbers are primes, 4, and 1. For 4, we see the base representations 1002 and 113, which is not possible.
6. Find the number of ordered pairs of integers \((x, y)\) such that \(2167\) divides \(3x^2 + 27y^2 + 2021\) with \(0 \leq x, y \leq 2166\). 

**Answer:** \(2352\)

First, we observe that \(2167 = 11 \cdot 197\), and so by Chinese Remainder Theorem we just determine the number of ways to do this for \(p = 11\) and \(p = 197\).

For \(p = 11\), this reduces down to the congruence \(3x^2 + 27y^2 \equiv 3 \pmod{11}\), or that \(x^2 + 9y^2 \equiv 1 \pmod{11}\). Since \(9\) is a square, we see that we can write \(z = 3y\) and solve \(x^2 + z^2 \equiv 1 \pmod{11}\), and get the same number of solutions (since we can then find \(y\) again given \(z\)).

As for \(p = 197\), we get that \(3x^2 + 27y^2 \equiv 51 \pmod{197}\), or that \(x^2 + 9y^2 \equiv 17 \pmod{197}\), which we may again write as \(x^2 + z^2 \equiv 17 \pmod{197}\). Notice, however, that \(197 \equiv 1 \pmod{4}\), meaning that \(17\) is a quadratic residue of \(197\) if and only if \(197\) is one of \(17\), or that \(10\) is a square \((\pmod{17})\). We can see that \(10^8\) \((\pmod{17}) \equiv (-2)^4 \equiv 1\) \((\pmod{17})\), meaning that, in fact, \(17\) is a non-quadratic residue \((\pmod{197})\).

We now claim that the first equation has 12 solutions, and the second has 196. Here let \(p = 197\) and \(r = 17\). Let the number of solutions be \(N\) for \(x^2 + z^2 \equiv r \pmod{p}\), where \(r \neq 0\). Then, we have \(N = \sum_{a+b=r}(1 + \left(\frac{a}{p}\right))(1 + \left(\frac{b}{p}\right))\). Thus \(N = p + \sum_a \left(\frac{a}{p}\right) + \sum_b \left(\frac{b}{p}\right) + \sum_{a+b=r}\left(\frac{a}{p}\right)\left(\frac{b}{p}\right)\). The first two sums are easily seen to be 0. As for the third one, we consider the possibilities that we’re allowed to have. First, suppose that \(a, b\) are both squares; notice then that, since \(197 \equiv 1 \pmod{4}\), \(-1\) is a square too, so we find the number of solutions to \((x - y)(x + y) = x^2 - y^2 \equiv r \pmod{197}\). Notice that, given \(x - y \neq 0\), we can find \(x + y\) and thus \(x, y\). This yields us with 196 solutions. But considering the signs that are allowed, we see that we can negate \(x, y\) freely, and since \(17\) isn’t a square modulo \(106\), but \(-1\) is, we can’t have either be 0, yielding us with \(\frac{p - 1}{2} = 99\) solutions here.

Therefore, since we have \(\frac{p + 1}{2} = 99\) squares, we thus have 50 pairs where \(a\) is a square, \(b\) isn’t, and so 50 where \(b\) is a square, \(a\) isn’t, and therefore 48 where neither are squares. However, notice that we have two terms, namely those with \((0, 17)\) and \((17, 0)\) that we subtract because they contribute 0, not 1. But then notice that we get \(49 - 48 - 50 - 48 - 2 = -1\). Therefore, we have that \(N = p - 1\).

We now run this argument for \(p = 11\). Notice that we end up getting that, for \(a + b = 1\), since \(-1\) is a nonquadratic residue for \(11\), we see that the number where \(a\) is a square, \(b\) isn’t is the number of solutions \(x^2 - y^2 = 1\), where \(y \neq 0\). We have in total 10 solutions for \(x, y\), of which 2 have \(y = 0\), and then we divide again by 4 to get 2 solutions in total. Thus, we have \(6 - 2 = 4\) pairs where both are squares, \(2\) again with one but not the other, and \(3\) where both are not squares. This then evaluates to 3. But again, here we have the pairs \((0, 1)\) and \((1, 0)\) which contribute 0 each, not 1, so we subtract 2. Therefore, we see that we have \(p + 4 - 2 - 2 + 3 - 2 = p + 1\) solutions here.

Our answer is thus \(12 \cdot 196 = 2352\).

7. Let \(\phi(x, v)\) be the smallest positive integer \(n\) so that \(2^n\) divides \(x^n + 95\) if it exists, or 0 if no such positive integer exists. Determine \(\sum_{i=0}^{255} \phi(i, 8)\).

**Answer:** \(2704\)

All equivalences here are \(\text{mod} 256\).

First, we observe that \(6561 + 95 \equiv 6656 = 256 \cdot 26 \equiv 0\), and \(6561 = 3^8\), so we can write the desired divisibility as \(2^n | x^n - 3^8\).
We now instead compute the number of $i$ such that $\phi(i, 8) = n$ for each $n > 0$. Write $n = b2^a$, where $b$ is odd.

First, we'll show that $a \leq 3$ for there to be at least one solution.

By continuing squaring, we see that $(-95)^2 = 65$, $65^2 = 129, 129^2 = 1$, which means that $3^{64} \equiv 1$, but $3^{32}$ is not equivalent to 1. But note that $x^{64} - 1 \equiv 0$ for all odd $x$, since writing $x = 2y + 1$ yields that $x^{64} - 1 \equiv 128(y + 63y^2) \equiv 0$. Thus, $x^{64} \equiv x^8$, with $a > 3$, implies that $1 \equiv 3^{2^{a-3}}$, contradiction with $a > 3$.

Now, we know that $a \leq 3$. Note that we expand out to get that we want $x$ so that $(x^3 - 3^2)(x^3 - 3^2 + 1) \cdots (x^{2^a-3} - 3^2 + 1)$. Note that none of the terms other than the first 1 can contribute a power of 2 that is larger than 2, since these terms will be equivalent to 2 mod 4. Note also that at most one of the first two terms can be divisible by 4.

If $a > 0$, then either $x^3 \equiv 3^{2^{a-2}} \mod 2^8$, or $x^3 \equiv -3^{2^{a-2}} \mod 2^8$. If $a = 0$, this is just $x^3 \equiv 3^8$.

But $b$ is odd, so it has an inverse modulo any power of 2. Raising each of these equations to their appropriate powers yields a unique solution modulo $2^8$.

Thus, the number of solutions for $n$ is 1 if $a = 0$ and $2^a + 1$ if $1 \leq a \leq 3$.

Now, say $x^m = x^n \equiv 3^b$. Write $m = y2^a, n = z2^a$, with $y, z$ odd. If $a \neq b$, WLOG $a < b$.

Then $x^{b-a} = 1$ gives that $x^{2^a(y-2^a)} \equiv 1$. But $2^{b-a}y - z$ would be odd, so we can raise this to $2^{b-a}y - z$'s inverse modulo 64, giving $x^{2^a} \equiv 1$, which means that $x^{2^a} \equiv 3^b \equiv 1$, a contradiction.

If $a = b$, repeating this yields that $x^{2^a(y-z)} \equiv 1$, or that $3^b(y-z)$, by raising to the $y$th power. But then we note that $y - z$ must be divisible by 8. Thus, we see that we have 16 possible values of $n : 1, 3, 5, 7, 2, 6, 10, 14, 12, 20, 28, 8, 24, 45, 56$.

Summing these yields the answer $(1 + 3 + 5 + 7)(1*1+2*4+4*4+8*8+8*16) = 16*(1+8+32+128) = 16*169 = 2704$.

8. What is the smallest integer $a_0$ such that, for every positive integer $n$, there exists a sequence of distinct positive integers $a_0, a_1, ..., a_{n-1}, a_n$ such that $a_0 = a_n$, and for $0 \leq i \leq n - 1$, $a_{i+1}$ ends in the digits $0a_i$ when expressed without leading zeros in base 10?

**Proposed by: Austen Mazenko**

**Answer:** 7

Evidently, $a_0$ must be relatively prime to 10. First, we note that $a_0 \neq 3$; if it were, then $3^{a_1} \equiv 3 \pmod{100}$, and since $\text{ord}_{100}(3) = 20$ we need $a_1 \equiv 1 \pmod{20}$. Furthermore, if $a_1$ has $k$ digits, we need $a_1^3 \equiv a_1 \pmod{10^{k+1}}$, so $(a_1 - 1)(a_1 + 1) \equiv 0 \pmod{10^{k+1}}$. Thus, $a_1 \equiv 1 \pmod{5^{k+1}}$, which combined with $a_1 \equiv 1 \pmod{4}$ means $\nu_2(a_1 + 1) = 1$. But, $\nu_2((a_1 - 1)(a_1 + 1)) \geq k + 1$ so $\nu_2(a_1 - 1) \geq k$. In particular, $a_1 - 1 \geq 2^k \cdot 5^{k+1} = 5 \cdot 10^k$, so $a_1$ has more than $k$ digits, contradiction.

Now we claim that $a_0 = 7$ works. If $n = 2$, then pick $a_1 = 2 \cdot 5^2 - 1 = 781249$. First, $\text{ord}_{100}(7) = 4$, and since 781249 $\equiv 1 \pmod{4}$ we have $781249^4 \equiv 7 \pmod{100}$. Then, $781249^2 \equiv 2 \cdot 5^2 \cdot (2 \cdot 5^2 - 2) + 1 \equiv 2^2 \cdot 5^4 \cdot (5^2 - 1) \equiv 1 \pmod{10^7}$ since $\nu_2(5^4 - 1) = 2 + 3 + 1 - 1 = 5$ by LTE. Hence, 781249$^9 \equiv (781249^2)^3 \equiv 1 \pmod{10^9}$, as desired.

Otherwise, consider the arbitrarily long sequence $a_0 = 7, a_k = 2 \cdot 10^{k+1} + 1, a_{n-1} = 74218751$ for $0 < k < n - 1$. First, $21 \equiv 1 \pmod{4}$ implies $721 \equiv 7 \pmod{100}$. Now, by the binomial theorem it is evident $(2 \cdot 10^{k+1})^{50} \equiv 1 \pmod{10^{k+2}}$, and because $2 \cdot 10^{k+1+1} \equiv 1 \pmod{50}$ for $k \geq 1$, we have $(2 \cdot 10^{k+1})^{2 \cdot 10^k + 1} \equiv 2 \cdot 10^k + 1 \pmod{10^{k+2}}$, and similarly for the exponent 74218751 $\equiv 1 \pmod{50}$. It remains to show 74218751$^2 \equiv 1 \pmod{10^9}$. We have 74218750 = 2 * 5^9 * 19 + 1,
so \(74218751^2 - 1 = 2 \cdot 5^9 \cdot 19 \cdot 2(5^9 \cdot 19 + 1)\), meaning we must show \(\nu_2(5^9 \cdot 19 + 1) \geq 7\). Now, 
\[5^3 \equiv -3 \pmod{2^7}\], so \(5^9 \equiv -27 \pmod{2^7}\), thus \(5^9 \cdot 19 + 1 \equiv -27 \cdot 19 + 1 \equiv -512 \equiv 0 \pmod{2^7}\), as desired.