Mathematical Aspects of GPS RAIM

Two Results on Integrity Monitoring and Fault Detection

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Two new theorems will answer two questions about RAIM and the least squares navigation solution.

(1) What can be achieved by RAIM from a snapshot of data?

(2) What is the relationship between the two nav solutions obtained by

   (a) excluding a biased measurement ("active" RAIM)

   (b) including the biased measurement but compensating for its effect using RAIM techniques ("passive" RAIM)
Background

Least Squares Navigation Solution

Error & Fault Vector

Standard measurement equation:
\[ z = Hx + e \]

Least squares solution:
\[ \hat{x} = H^*z \]
where \[ H^* = (H^TH)^{-1}H^T \]

Navigation error:
\[ d := \hat{x} - x \]
\[ = H^*z - x \]
\[ = H^*(Hx + e) - x \]
\[ = H^*e \]

Fault vector:
\[ f := Sz \]
where \[ S := I - HH^* \]
\[ f := S(Hx + e) \]
\[ = Se \]
Theorem 1: Independence of \( f \) and \( d \)

Question:
Given a snapshot of measurements, is it ever possible to determine the navigation error from the fault vector?

Answer:
No.

Theorem:
Consider the measurement equation, \( f \), and \( d \).

\[
\begin{align*}
    z &= Hx + e \\
    f &= Sz \\
    d &= \hat{x} - x = H^e
\end{align*}
\]

If the elements of \( e \) are IID random variables, then \( f \) is statistically independent of \( d \).
Theorem:
\[ e \text{ IID } \Rightarrow f \text{ is independent of } d. \]

Proof:

\( \text{svd of } H: \)

\[ H = U \begin{bmatrix} \Sigma \\ 0 \end{bmatrix} V^T \]

\[ H^* = V [ \Sigma^{-1} 0 ] U^T \]

\[ S = I - HH^* \]

\[ = U \begin{bmatrix} 0 \\ 1 \end{bmatrix} [ 0 \ 1 ] U^T \]

\[ f = Se \]

\[ = U \begin{bmatrix} 0 \\ 1 \end{bmatrix} [ 0 \ 1 ] U^T e \]

\[ d = H^*e \]

\[ = V [ \Sigma^{-1} 0 ] U^T e \]

\[ \tilde{e} := \begin{bmatrix} \tilde{e}_1 \\ \tilde{e}_2 \end{bmatrix} := U^T e \]

\[ f = U \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tilde{e}_2. \quad d = \Sigma^{-1} \tilde{e}_1 \]
Background Math, Active RAIM
Fault (Parity) Vector & Error Estimate

Active RAIM; measurement equation:

\[ z_{0i} = H_{0i}x + c_{0i} \]

Fault vector:

\[ f = S_\varepsilon \]
\[ f_i = S_{i*}e \]

Error model:

\[ \hat{e} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \hat{e}_i \\ 0 \\ \vdots \\ 0 \end{bmatrix} \]

Bias estimate:

\[ \hat{e}_i = \frac{f_i}{S_{ii}} \]

Nav error estimate:

\[ \hat{d} = H^\star \hat{e} \]
\[ = H^\star_i \hat{e}_i \]
\[ = H^\star_i \frac{S_{i*}e}{S_{ii}} \]
Theorem 2: Equivalence of active & passive RAIM

Consider the three navigation solutions

\[ \hat{x} := H^* z \quad \text{biased nav solution} \]
\[ \hat{x}_{RAIM} := \hat{x} - \hat{d} \quad \text{compensated solution (passive RAIM)} \]
\[ \hat{x}_{0i} := H_{0i}^* z_{0i} \quad \text{unbiased solution (active RAIM)} \]

Question:
What is the relationship between \( \hat{x}_{RAIM} \) and \( \hat{x}_{0i} \)?

Answer:
They are identical.

Theorem:
\[ \hat{x}_{RAIM} = \hat{x}_{0i} \]

Proof needs a lemma and its corollary.

Lemma:
\[ H_{0i}^* = H^* - H^* \frac{S_i^*}{S_{ii}} \]

Corollary:
The \( i \)th column of \( H_{0i}^* \) equals zero.
Theorem: \[ \hat{x}_{RAIM} = \hat{x}_{0i} \]

Proof:

\[ \hat{x}_{RAIM} := \hat{x} - \hat{d} \]
\[ = H^* z - H^* i \frac{S_i^i}{S_{ii}} e \]
\[ = H^* (Hx + e) - H^* i \frac{S_i^i}{S_{ii}} e \]
\[ = x + (H^* - H^* i \frac{S_i^i}{S_{ii}}) e \]

\[ \hat{x}_{0i} = H_{0i}^* z_{0i} \]
\[ = H_{0i}^* (H_{0i} x + e_{0i}) \]
\[ = x + H_{0i}^* e_{0i} \]
\[ = x + H_{0i}^* e \]

Recall Lemma:

\[ H_{0i}^* = H^* - H^* i \frac{S_i^i}{S_{ii}} \]

Using the lemma, the coefficients of \( e \) are equal.
Example
(from the GPS Toolbox for Matlab)

» help raimdemo
  raimdemo
  function to run lsnav.m with some real data collected at Wallops Island, VA, USA
  The user has the option of corrupting one of the pseudo-ranges with a bias,
  raim.m is then run to try and detect the bias and its effect on the navigation solution.
  This demo illustrates the use of raim.m with lsnav.m

» raimdemo
There are 8 satellite measurements, from satellites:
  2, 7, 13, 14, 15, 19, 27, 31
With elevations (in degrees): 71, 49, 21, 20, 41, 16, 37, 5
Select a satellite to bias :2
Select the bias in meters :100
A bias of 100m has been added to satellite 2
Navigating, using only the 7 unbiased satellites

.............................................................
Navigating using all 8 satellites, with RAIM to detect bias and correct navigation error

.............................................................
Hit any key to continue
Correcting the biased solution with the RAIM estimate of the error

» navsys
Example
(from the GPS Toolbox for Matlab)

Altitude of aircraft vs time, Unbiased solution (—) and biased solution (—)

Altitude vs time. Unbiased solution (—) & RAIM corrected biased solution (+)

raim.m found the correct bias 100% of the time, Bhat = 115.451m
Example
(from USCG R&D Field Test)
Conclusion

Theorem 1:
It is essential to take an ensemble of measurements to predict the precision of the nav solution.

Theorem 2:
If a biased measurement is detected, it is just as good to correct the biased nav solution as to exclude the measurement before making the nav solution.