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Debt, deficits and finite horizons: The stochastic case

Roger E.A. Farmer [a,*], Carine Nourry [b], Alain Venditti [c]

[a] University of California, Los Angeles, United States
[b] Université de la Méditerranée, Institut Universitaire de France & GREQAM, France
[c] CNRS-GREQAM & EDHEC, France

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ABSTRACT

We introduce aggregate uncertainty and complete markets into Blanchard's (1985) perpetual youth model. We derive a simple expression for the pricing kernel that can be used to close a variety of equilibrium models in which the set of agents changes over time.

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1. Introduction

The overlapping generations model has very different properties from that of the representative agent model but it does not lend itself to empirical work because realistically calibrated versions of the model are described by very high order difference equations. To circumvent this problem, Olivier Blanchard (1985) and Philippe Weil (1989) have studied a model populated by a measure of long-lived agents that die, and are replaced, with a fixed probability that is independent of age. This *perpetual youth model* combines features of the representative agent model with the OG framework in a tractable way and versions of the model have been used to study a variety of issues in macroeconomics.

The device of assuming long-lived agents who die with fixed probability is a useful one because it allows the researcher to construct tractable models in which Ricardian equivalence (see Barro (1974)) breaks down and fiscal transfers have real effects. But although the perpetual youth model is tractable, the versions that have been worked out in the existing literature, do not allow for aggregate shocks in an internally consistent way.[1] In this paper we show how to construct the pricing kernel in models with aggregate uncertainty under the assumption of complete markets. Our result can be applied to close a variety of models ranging from endowment economies to monetary or non-monetary versions of production economies.

2. Population dynamics

We assume that a new cohort of individuals is born each period. Agents die with fixed probability which is independent of age. This important assumption implies that all agents discount the future in the same way and it leads to a single concept of aggregate human wealth that greatly simplifies the problem of finding a simple expression for the pricing kernel.

Each household survives into the subsequent period with a fixed probability π and every period a proportion $(1-\pi)$ of households dies. At the beginning of each period, households have n children. It follows that if N_t is the number of agents alive at date t then

$$N_{t+1} = (\pi + n)N_t, \tag{1}$$

is the number of agents alive at date $t+1$. Depending on whether $\pi+n$ is greater or smaller than one, the total population will increase or decrease over time. We assume that $\pi+n>1$ and we normalize the initial population to one, $N_0=1$.

The combination of birth and death processes implies that at any point in time there are $(\pi+n)^t$ agents alive of whom $\pi(\pi+n)^{t-1}$ are

* Corresponding author. Tel.: +1 310 825 6547; fax: +1 310 825 9528.
E-mail address: rfarmer@econ.ucla.edu (R.E.A. Farmer).

[1] See for example, the papers by Ghironi (2006); Ganelli (2003, 2005) and Botman et al. (2006).

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"old", i.e. survivors from the previous period $t-1$, and $n(\pi+n)^{t-1}$ are "young",i.e. newly born in period t.

3. Aggregate uncertainty

Uncertainty of each period is indexed by a finite set of states $\mathbf{S}=\{S_1,\ldots S_n\}$. Define the set of t-period histories $\mathbf{S}^t$ recursively as follows:

$$\begin{aligned}\mathbf{S}^1 &= \mathbf{S}\\ \mathbf{S}^t &= \mathbf{S}^{t-1}\times\mathbf{S},\ t=2,\ldots\end{aligned} \tag{2}$$

We will use S to denote a generic element of $\mathbf{S}$ realized at date t and S' to denote an element of $\mathbf{S}$ realized at $t+1$.

The households trade a complete set of Arrow securities. Let $Q_t^\tau(S^\tau)$ represent the price of the security that pays one unit of the consumption commodity if and only if history $S^\tau\in\mathbf{S}^\tau$ occurs at date τ. Let $Q_t^{t+1}(S')$ be the price of an Arrow security. This is a claim, sold at date t, to one unit of the consumption good for delivery at date $t+1$ if and only if state S' occurs. Let the probability that S' occurs at date $t+1$ be given by $p(S')$ and assume that this probability is independent of time.

4. Annuities and the household budget constraint

For each period t, the agents of household i receive an income $w_t^i(S)$ if state $S\in\mathbf{S}$ is realized. They purchase consumption commodities $c_t^i(S)$ and they accumulate a portfolio of the n securities $a_{t+1}^i(S')$, where there is one security for each of the values of S'.

Since the household may not survive into period $t+1$, we assume, as in Blanchard (1985), that there exists an actuarially fair annuities market. The existence of this market implies that the household pays price $\pi Q_t^{t+1}(S')$ for a claim to one unit of consumption in the period $t+1$ if and only if state S' occurs and the household is alive. We assume that this security is issued by a competitive annuity sector that earns zero profit in equilibrium. If the household dies, its claim reverts to the company that issued the annuity.

Given our assumptions, the representative family born in period h faces the following sequence of budget constraints,

$$\sum_{S'\in\mathbf{S}}\pi Q_t^{t+1}(S')a_{t+1}^i(S') = a_t^i(S)+w_t^i(S)-c_t^i(S),\quad t=h\ldots\infty, \tag{3}$$

together with the set of no-Ponzi scheme conditions,

$$\lim_{T\to\infty}\pi^{T-h}Q_h^T\left(S^T\right)a_h^i\left(S^T\right)\geq 0,\ \text{for all}\ S^T\in\mathbf{S}^T, \tag{4}$$

one for every possible history that might occur. These constraints imply that the household must plan to remain solvent in every possible history. The term $a_{t+1}^i(S')$ is the quantity of security S' purchased for price $\pi Q_t^{t+1}(S')$ at date t. The terms $a_t^i(S)$ and $w_t^i(S)$ on the right side of Eq. (3) are respectively the sole security that has positive value at date t and the endowment received at date t if state S is realized. $c_t^i(S)$ is the household's purchase of consumption commodities.

5. The definition of human wealth

The human wealth of household i is defined recursively by the expression,

$$h_t^i(S) = w_t^i(S)+\sum_{S'\in\mathbf{S}}\pi Q_t^{t+1}(S')h_{t+1}^i(S'). \tag{5}$$

We assume that human wealth is finite and to guarantee this we impose the following condition,

$$\lim_{T\to\infty}\pi^{T-h}Q_h^T\left(S^T\right)w_h^i\left(S^T\right)<\infty,\ \text{for all}\ S^T\in\mathbf{S}^T. \tag{6}$$

Iterating Eq. (5) we write human wealth as an infinite sum,

$$h_t^i(S) = \sum_{\tau=t}^{\infty}\left[\pi^{\tau-t}\sum_{S^\tau\in\mathbf{S}^\tau}Q_t^\tau(S^\tau)w_\tau^i(S^\tau)\right]. \tag{7}$$

In a representative agent model with no uncertainty and a constant growth rate of endowments, a condition like Eq. (6) would imply that the interest rate must exceed the growth rate. Our condition is weaker. In the perpetual youth model the household values future assets using the factor πQ_t^{t+1} instead of Q_t^{t+1} reflecting the fact that it may not survive into the subsequent period. This fact implies that equilibrium models that are populated by the agents we describe in this paper may display inefficient equilibria in which the interest rate is less than the growth rate.

Eqs. (3)–(5) can be combined to write a single budget constraint for the household,

$$\sum_{\tau=t}^{\infty}\left[\pi^{\tau-t}\sum_{S^\tau\in\mathbf{S}^\tau}Q_t^\tau(S^\tau)c_\tau^i(S^\tau)\right]\leq h_t^i(S)+a_t^i(S). \tag{8}$$

6. The household problem

The per-period utility function of the agents is assumed to be isoelastic. For a typical agent i, utility at date t is given by the expression

$$U_t\left(c_t^i\right)=\begin{cases}\dfrac{\left(c_t^i\right)^{1-\rho}}{1-\rho} & \text{if}\ \rho>0,\rho\neq 1,\\ \log\left(c_t^i\right) & \text{if}\ \rho=1.\end{cases} \tag{9}$$

A representative family maximizes the following sum of discounted utilities

$$\mathbf{E}_t\left\{\sum_{\tau=t}^{+\infty}(\pi\beta)^{\tau-t}U\left[c_\tau^i(S^\tau)\right]\right\} \tag{10}$$

subject to the budget constraint (Eq. (8)), where $\beta\in(0,1)$ is the discount factor. The first order condition for this maximization program is given by the following set of equations, one for each history $S^\tau\in\mathbf{S}^\tau$,

$$\beta^{\tau-t}p(S^\tau)\left[c_\tau^i(S^\tau)\right]^{-\rho}=\lambda Q_t^\tau(S^\tau), \tag{11}$$

where λ is the multiplier on the budget constraint (Eq. (8)). This constraint must hold with equality at the optimum. Taking the ratio of Eq. (11) in two consecutive states leads to the expression,

$$c_t^i(S)=c_{t+1}^i(S')\left[\frac{\beta p(S')}{Q_t^{t+1}(S')}\right]^{\frac{-1}{\rho}}. \tag{12}$$

The term π does not appear in this expression since it appears both in the utility function and in the budget constraint and the two terms cancel each other out.

Recall that $p(S^\tau)$ is the probability that history S^τ occurs. With an isoelastic utility function, the solution to the household's problem is given by the following policy function[2]

$$c_t^i(S)=X_t^{-1}\left[a_t^i(S)+h_t^i(S)\right], \tag{13}$$

[2] The proof of this statement is standard and is available from the authors on request.

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where X_t is defined by the recursion

$$X_t = 1 + \pi(\beta)^{1/\rho} E_t \left[\frac{p(S')}{Q_t^{t+1}(S')} \right]^{\frac{1-\rho}{\rho}} X_{t+1}. \tag{14}$$

Eq. (13) instructs the household, in the optimal plan, to consume a fraction X_t^{-1} of wealth each period where X_t depends on all future prices. In the special case of logarithmic utility Eq. (13) simplifies to,

$$c_t^i(S) = (1 - \pi\beta)\left[a_t^i(S) + h_t^i(S)\right]. \tag{15}$$

7. Defining aggregate variables

Let $\mathcal{A}_t$ be the index set of all agents that are alive at date t. Recall that a proportion π of these agents will survive into period $t+1$. Similarly, let $\mathcal{N}_{t+1}$ denote the set of newborns at period $t+1$. For any date $t+1$ and any variable x let x_t^i be the quantity of that variable held by household i and let x_t be the aggregate quantity. Then,

$$\pi \sum_{i \in \mathcal{A}_t} x_{t+1}^i + \sum_{i \in \mathcal{N}_{t+1}} x_{t+1}^i = \sum_{i \in \mathcal{A}_{t+1}} x_{t+1}^i = x_{t+1}. \tag{16}$$

Notice that $\mathcal{A}_{t+1} \neq \mathcal{A}_t \cup \mathcal{N}_{t+1}$ as $(1-\pi)N_t$ agents die at the end of period t while nN_t agents are born at the beginning of period $t+1$.

Eq. (16) says that any aggregate variable x can be defined as the sum over groups of people in two different ways. We can add up x^i over everyone who was alive yesterday and add it to the sum over everyone who is born today. Or we can add up x^i over everyone who is alive today. Using this notation, define A_t, W_t, C_t and H_t as follows,

$$A_t = \sum_{i \in \mathcal{A}_t} a_t^i, \quad W_t = \sum_{i \in \mathcal{A}_t} w_t^i, \quad C_t = \sum_{i \in \mathcal{A}_t} c_t^i, \quad H_t = \sum_{i \in \mathcal{A}_t} h_t^i. \tag{17}$$

We also define each of these variables in per capita terms,

$$a_t = \frac{A_t}{N_t}, \quad w_t = \frac{W_t}{N_t}, \quad c_t = \frac{C_t}{N_t}, \quad h_t = \frac{H_t}{N_t}, \tag{18}$$

where N_t evolves according to Eq. (1).

8. Deriving the pricing kernel

This section contains our main result. We show how to derive an expression for the pricing kernel in the perpetual youth model in terms of two observable variables: aggregate consumption and aggregate wealth.

We begin with Eq. (12), the agent's Euler equation. Aggregating this equation over the set $\mathcal{A}_t$ of agents that are alive at date t, we obtain the expression

$$\left[\frac{\beta p(S')}{Q_t^{t+1}(S')} \right]^{\frac{-1}{\rho}} \sum_{i \in \mathcal{A}_t} c_{t+1}^i(S') = \sum_{i \in \mathcal{A}_t} c_t^i(S). \tag{19}$$

The right hand side of this equation is equal to aggregate consumption,

$$\sum_{i \in \mathcal{A}_t} c_t^i = C_t. \tag{20}$$

To find a simple equation for the pricing kernel, we need an expression for $\sum_{i \in \mathcal{A}_t} c_{t+1}^i(S')$ in terms of observables. We will use two facts. First, the consumption of all agents is the same linear function of their wealth. Second, every aggregate variable can be split into a sum over new born agents and existing ones.

Using the aggregation definition, Eq. (16), and market clearing, it follows that

$$C_{t+1}(S') = \sum_{i \in \mathcal{A}_{t+1}} c_{t+1}^i(S') = \pi \sum_{i \in \mathcal{A}_t} c_{t+1}^i(S') + \sum_{i \in \mathcal{N}_{t+1}} c_{t+1}^i(S'). \tag{21}$$

Recall that newborns do not own Arrow securities. Their wealth is in the form of human wealth. Using the policy function (Eq. (13)) evaluated at date $t+1$ and the fact that

$$C_{t+1} = X_{t+1}^{-1}\left[A_{t+1} + H_{t+1}\right], \tag{22}$$

we can find an expression for the newborn consumption in terms of aggregate wealth and aggregate consumption both of which are observable. Using the additional fact that the fraction of newborns in the population is $n/(n+\pi)$ we can write this expression as follows,

$$\sum_{i \in \mathcal{N}_{t+1}} c_{t+1}^i(S') = \frac{n}{\pi + n}\left[C_{t+1} - X_{t+1}^{-1} A_{t+1}\right]. \tag{23}$$

Combining Eqs. (19)–(23) leads to the expression

$$\left[\frac{\beta p(S')}{Q_t(S')} \right]^{\frac{-1}{\rho}} = \frac{\pi(\pi + n) C_t(S)}{\left[\pi C_{t+1}(S') + n A_{t+1} X_{t+1}^{-1}\right]}. \tag{24}$$

Dividing top and bottom by N_t and rearranging terms give an expression for $Q_t(S')$ in terms of the per capita variables $c_t(S)$ and $a_t(S)$.

$$Q_t(S') = \beta p(S') \left[\frac{\pi c_t(S)}{\pi c_{t+1}(S') + n a_{t+1}(S') X_{t+1}^{-1}(S')} \right]^{\rho} \tag{25}$$

is our main result. Combined with Eq. (14), which we reproduce below, this equation can be combined with an aggregate model for consumption and be used to price assets.

$$X_t = 1 + \pi(\beta)^{1/\rho} E_t \left[\frac{p(S')}{Q_t^{t+1}(S')} \right]^{\frac{1-\rho}{\rho}} X_{t+1}. \tag{26}$$

A model based on this structure could be closed in a number of ways. In an endowment economy consumption c_t would be set equal to the exogenous per capita aggregate endowment ω_t. One could assume that aggregate assets a_t equal zero or that government issues money or debt that equals a_t in equilibrium. A version with production would set a_t equal to the capital stock and it would add an equation to determine Q_t in each state as a function of the marginal product of capital.

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