American Economic Review 2010, 100:1, 608–617
http://www.aeaweb.org/articles.php?doi=10.1257/aer.100.1.608

Generalizing the Taylor Principle: Comment

By Roger E. A. Farmer, Daniel F. Waggoner, and Tao Zha*

In a recent paper in this journal, Troy Davig and Eric M. Leeper (2007) provide a generalization of the Taylor principle, the proposition that central banks can stabilize the economy by raising their interest rate instrument more than one for one in response to higher inflation. The Taylor principle was originally discussed in the context of a constant parameter new Keynesian model.[1] Davig and Leeper generalize this principle to a class of forward-looking Markov switching rational expectations models. Reduced-form Markov switching models have been popular tools for studying a wide range of economic issues (Christopher Sims and Tao Zha 2006), including changes in the monetary transmission mechanism, and the Davig-Leeper extension of the Taylor principle to forward looking environments is important since it permits the analysis of alternative policies within a rational expectations framework.

I. The Regime Switching Model

The new Keynesian model analyzed by Davig and Leeper has two private sector equations,

$$x_t = E_t x_{t+1} - \sigma^{-1}(i_t - E_t \pi_{t+1}) + u_t^D \text{ and} \tag{1}$$

$$\pi_t = \beta E_t \pi_{t+1} + \kappa x_t + u_t^S, \tag{2}$$

where x_t is output, π_t is inflation, i_t is the nominal interest rate, u_t^D is an aggregate demand shock, and u_t^S is an aggregate supply shock. The variables π_t and i_t are percentage deviations from their steady-state values and x_t is the deviation of output from its trend path. Davig and Leeper assume that u_t^D and u_t^S are bounded first-order autoregressive processes. To derive closed-form solutions, we simplify their example by setting the autoregressive coefficients to zero, which implies that u_t^D and u_t^S are bounded, mean zero, and i.i.d. random variables. Nothing of substance hinges on this assumption and it has the advantage of simplifying our notation.

The policy rule is given by

$$i_t = \alpha_{s_t} \pi_t + \gamma_{s_t} x_t, \tag{3}$$

where s_t is a two-regime Markov process assuming values in $\{1,2\}$ with transition matrix $\mathbf{P} = (p_{ij})$ for $i, j = 1, 2$, with p_{ij} being the probability that $s_t = j$, given that $s_{t-1} = i$. As in Davig and Leeper, we assume that the fundamental shocks u_t^D and u_t^S are independent of the Markov process s_t.

* Farmer: UCLA Department of Economics, 8283 Bunche Hall, Box 951477, Los Angeles, CA 90095-1477, NBER, and CEPR (e-mail: rfarmer@econ.ucla.edu); Waggoner: Federal Reserve Bank of Atlanta, 1000 Peachtree Street NE, Atlanta, GA 30309-4470 (e-mail: dwaggoner@frbatlanta.org); Zha: Federal Reserve Bank of Atlanta, 1000 Peachtree Street NE, Atlanta, GA 30309-4470, and Emory University (e-mail: tzha@frbatlanta.org). We thank the referees of this journal and Mark Gertler for helpful comments. The views expressed herein do not necessarily reflect those of the Federal Reserve Bank of Atlanta nor those of the Federal Reserve System. Farmer acknowledges the support of National Science Foundation grant SBR 0720839.

[1] See, for example, Michael Woodford (2003) and the references therein.

The private sector block, consisting of equations (1) and (2), has three regime-independent parameters, σ, β, and κ. Uncertain monetary policy is represented by equation (3), the policy rule. This equation has two regime-dependent parameters, α_{s_t} and γ_{s_t}, which capture the degree to which monetary policy responds to inflation and output. By substituting the policy rule (3) into equation (1) and rearranging the terms, the regime switching new Keynesian model can be written in matrix form as

$$\mathbf{\Gamma}_{s_t}\mathbf{y}_t = E_t\mathbf{y}_{t+1} + \mathbf{\Psi}\mathbf{u}_t, \tag{4}$$

where

$$\mathbf{y}_t = \begin{bmatrix} \pi_t \\ x_t \end{bmatrix}, \ \mathbf{u}_t = \begin{bmatrix} u_t^S \\ u_t^D \end{bmatrix}, \tag{5}$$

$$\mathbf{\Gamma}_{s_t} = \begin{bmatrix} \beta^{-1} & -\kappa\beta^{-1} \\ \sigma^{-1}(\alpha_{st} - \beta^{-1}) & 1 + \sigma^{-1}(\gamma_{st} + \kappa\beta^{-1}) \end{bmatrix}, \text{ and } \mathbf{\Psi} = \begin{bmatrix} \beta^{-1} & 0 \\ -\sigma^{-1}\beta^{-1} & 1 \end{bmatrix}. \tag{6}$$

II. The Taylor Principle in a Constant Parameter Model

Before explaining the regime switching model, we present a brief analysis of the role of the Taylor principle in the context of the constant parameter model. Although this model is well known, we review it because the properties of indeterminate equilibria in this familiar context are essential to clarifying the results we will present for the regime switching case.

The constant parameter new Keynesian model is given by the equation

$$\mathbf{\Gamma}\,\mathbf{y}_t = E_t y_{t+1} + \mathbf{\Psi}\mathbf{u}_t, \tag{7}$$

where

$$\mathbf{\Gamma} = \begin{bmatrix} \beta^{-1} & -\kappa\beta^{-1} \\ \sigma^{-1}(\alpha - \beta^{-1}) & 1 + \sigma^{-1}(\gamma + \kappa\beta^{-1}) \end{bmatrix}.$$

A solution to (7) is a stochastic process, satisfying this equation, which describes how the vector of variables y_t evolves through time. Depending on the values of the parameters, there may be one or more solutions. One solution describes y_t as a linear function of the shocks:

$$\mathbf{y}_t = \mathbf{G}\mathbf{u}_t, \tag{8}$$

where

$$\mathbf{G} = \mathbf{\Gamma}^{-1}\mathbf{\Psi} = \frac{1}{\sigma + \gamma + \kappa\alpha}\begin{bmatrix} \sigma + \gamma & \kappa\sigma \\ -\alpha & \sigma \end{bmatrix}. \tag{9}$$

Following Bennet McCallum (1983), we refer to equation (9) as a minimal state variable solution. Since the fundamental shock $\mathbf{u}_t$ is bounded, the minimal state variable solution is also bounded in the sense that there exists a real number N such that

$$\|\mathbf{y}_t\| < N, \text{ for all } t,$$

where $|| : ||$ is any well-defined norm. This solution is unique in the class of bounded solutions if and only if all the roots of $\mathbf{\Gamma}$ are outside the unit circle.

Davig and Leeper focus on bounded solutions because local approximations to the underlying dynamic stochastic general equilibrium (DSGE) model remain valid in the neighborhood of a steady state, and the boundedness assumption allows one to appeal to approximation theorems that assert that the linear model is approximately valid for small noise. Throughout this paper, we follow Davig and Leeper and consider only solutions that remain bounded in this sense.

For some parameter configurations there may be an infinite set of solutions to equation (7). When this occurs, each member of the set is said to be an indeterminate equilibrium. The minimal state variable solution is a member of this set but there are other solutions that are serially correlated and add additional volatility to the time paths of the interest rate, output, and inflation. In recent papers on the empirical importance of indeterminate equilibria, Thomas Lubik and Frank Schorfheide (2003, 2004) show how to write an indeterminate solution to the constant parameter new Keynesian model as a linear combination of the minimal state variable solution and a first-order moving average component. When there are eigenvalues of $\mathbf{\Gamma}$ that are less than one in absolute value, the Lubik-Schorfheide indeterminate *fundamental* solutions to equation (7) have the form[2]

$$\mathbf{y}_t = \mathbf{G}u_t + \mathbf{V}\mathbf{w}_t, \tag{10}$$

$$\mathbf{w}_t = \mathbf{\Lambda} w_{t-1} + \mathbf{M}u_t, \tag{11}$$

where the random variable $\mathbf{w}_t$ is k-dimensional, $\mathbf{M}$ is any $k \times 2$ real matrix, $\mathbf{V}$ is $2 \times k$, $\mathbf{\Lambda}$ is $k \times k$, and k is the number of eigenvalues of $\mathbf{\Gamma}$ that are less than one in absolute value. The matrices $\mathbf{V}$ and $\mathbf{\Lambda}$ (obtained from the Schur decomposition of $\mathbf{\Gamma}$) satisfy the condition

$$\mathbf{V\Lambda} = \mathbf{\Gamma V}. \tag{12}$$

By direct substitution one can verify that our proposed solution, represented by equations (10) and (11), does indeed satisfy equation (7). The following steps establish this result:

$$E_t y_{t+1} + \mathbf{\Psi u}_t = E_t[\mathbf{G}u_{t+1} + \mathbf{V}(\mathbf{\Lambda w}_t + \mathbf{M}u_{t+1})] + \mathbf{\Psi u}_t \tag{13}$$

$$= \mathbf{V\Lambda w}_t + \mathbf{\Psi u}_t \tag{14}$$

$$= \mathbf{\Gamma V w}_t + \mathbf{\Gamma G u}_t \tag{15}$$

$$= \mathbf{\Gamma y}_t. \tag{16}$$

Equation (13) substitutes equations (10) and (11), led by one period, into the right-hand side of equation (7). Taking expectations and using the property that u_{t+1} is a mean zero random variable leads to equation (14). Equation (15) follows from equations (12) and (9), and equation (16) is established by collecting terms in $\mathbf{\Gamma}$ and applying the definition of a solution, equation (10).

It is worth drawing attention to two special cases of the Lubik-Schorfheide solutions. First, if all the eigenvalues of $\mathbf{\Gamma}$ are greater than one in absolute value, the minimal state variable

[2] Although our analysis applies to equilibria with both fundamental and nonfundamental components, we focus throughout this paper on indeterminate *fundamental* equilibria by setting the sunspot shock to zero. These bubble-free solutions represent equilibrium responses of agents to economically relevant fundamental shocks.

solution, equation (8), is unique. A special case of the new Keynesian model occurs if $\gamma = 0$. This case has been important in the literature since it leads to a particularly simple statement of the uniqueness condition, which is satisfied if $|\alpha| > 1$. When the central bank follows a policy of this kind in which it changes the interest rate by more than one for one in response to a change in inflation, the central bank is said to follow the Taylor principle.

A second important special case occurs when $\mathbf{\Gamma}$ has only one eigenvalue less than one in absolute value. In this case, $\mathbf{\Lambda}$ is equal to this eigenvalue, and $\mathbf{V}$ is its associated eigenvector. An example that has this property is provided by the parameterization $\beta = 0.99$, $\sigma = 1.0$, $\kappa = 0.17$, $\gamma = 0.0$, and $\alpha = 0.92$, For these parameters, the matrix $\mathbf{\Gamma}$ is given by

$$\mathbf{\Gamma} = \begin{bmatrix} 1.0101 & -0.1717 \\ -0.0901 & 1.1717 \end{bmatrix}, \tag{17}$$

which has real roots of 1.2392 and 0.9426. The eigenvector associated with the eigenvalue of 0.9426 is

$$\begin{bmatrix} -0.9306 \\ -0.3659 \end{bmatrix} \tag{18}$$

and the indeterminate solution is

$$\mathbf{y}_t = \begin{bmatrix} 0.8648 & 0.1470 \\ -0.7956 & 0.8648 \end{bmatrix} \mathbf{u}_t + \begin{bmatrix} -0.9306 \\ -0.3659 \end{bmatrix} \mathbf{w}_t, \tag{19}$$

where

$$\mathbf{w}_t = 0.9426\mathbf{w}_{t-1} + \mathbf{M}\mathbf{u}_t. \tag{20}$$

This example is important because Lubik and Schorfheide (2004) have documented that US monetary policy in the period from 1960 to 1979 did not satisfy the Taylor principle and that, as a consequence, the data were generated by an indeterminate equilibrium of this form.

III. The Generalized Taylor Principle in a Regime Switching Model

In this section we briefly describe Davig and Leeper's generalized Taylor principle. Analogous to the constant parameter model, the regime switching model also has a minimal state variable solution,

$$y_t = \mathbf{G}_{s_t}\mathbf{u}_t, \tag{21}$$

where

$$\mathbf{G}_{s_t} = \mathbf{\Gamma}_{s_t}^{-1}\mathbf{\Psi} = \frac{1}{\sigma + \gamma_{s_t} + \kappa\alpha_{s_t}} \begin{bmatrix} \sigma + \gamma_{s_t} & \kappa\sigma \\ -\alpha_{s_t} & \sigma \end{bmatrix}. \tag{22}$$

Since the fundamental shocks are bounded, so is the minimal state variable solution.

The Taylor principle provides a simple rule to ensure uniqueness of equilibrium: it works by guaranteeing that all the roots of a given matrix lie inside the unit circle. One would like to find a similar condition to establish a region of the parameter space for which the minimal state variable solution to equation (4) is the unique bounded equilibrium. This is a challenging problem since the parameters of the model are functions of the switching variable s_t, and these parameters enter multiplicatively with y_t. As a consequence, the regime switching model is inherently nonlinear.

To address the nonlinearity of the model, Davig and Leeper introduce additional variables that coincide with the original variables in some regimes, and they study an expanded model that is linear in these newly defined objects. The new variables are $\pi_{1,t}$, $\pi_{2,t}$, $x_{1,t}$, and $x_{2,t}$, which are random variables with the property

$$\pi_t = \begin{cases} \pi_{1,t} & \text{if } s_t = 1 \\ \pi_{2,t} & \text{if } s_t = 2 \end{cases} \text{ and } x_t = \begin{cases} x_{1,t} & \text{if } s_t = 1 \\ x_{2,t} & \text{if } s_t = 2 \end{cases}. \tag{23}$$

With these new random variables, Davig and Leeper derive the following linear system from the original Markov switching model equation (4):[3]

$$\mathbf{B}\mathbf{Y}_t = \mathbf{A}E_t Y_{t+1} + \mathbf{C}\mathbf{u}_t, \tag{24}$$

where

$$\mathbf{Y}_t = \begin{bmatrix} \pi_{1,t} \\ \pi_{2,t} \\ x_{1,t} \\ x_{2,t} \end{bmatrix}, \ \mathbf{A} = \begin{bmatrix} \beta p_{11} & \beta p_{12} & 0 & 0 \\ \beta p_{21} & \beta p_{22} & 0 & 0 \\ \sigma^{-1} p_{11} & \sigma^{-1} p_{12} & p_{11} & p_{12} \\ \sigma^{-1} p_{21} & \sigma^{-1} p_{22} & p_{21} & p_{22} \end{bmatrix}, \tag{25}$$

$$\mathbf{B} = \begin{bmatrix} 1 & 0 & -\kappa & 0 \\ 0 & 1 & 0 & -\kappa \\ \sigma^{-1}\alpha_1 & 0 & 1 + \sigma^{-1}\gamma_1 & 0 \\ 0 & \sigma^{-1}\alpha_2 & 0 & 1 + \sigma^{-1}\gamma_2 \end{bmatrix}, \text{ and } \mathbf{C} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}. \tag{26}$$

Multiplying equation (24) by A^{-1} transforms this equation into the form of equation (7), where $\mathbf{\Gamma} = \mathbf{A}^{-1}\mathbf{B}$ and $\mathbf{\Psi} = \mathbf{A}^{-1}\mathbf{C}$. It follows from the results discussed in Section II that equation (24) has a unique bounded solution if and only if all the eigenvalues of $\mathbf{\Gamma} = \mathbf{A}^{-1}\mathbf{B}$ are greater than one in absolute value. This is an application of Davig and Leeper's generalized Taylor principle to the case when the matrix $\mathbf{A}$ is nonsingular. More generally, the principle asserts that all the generalized eigenvalues of $(\mathbf{B}, \mathbf{A})$ lie inside the unit circle.

Davig and Leeper show, using Definition (23), that a solution to equation (24) can be used to construct a solution to the original nonlinear system, equation (4): but this does not establish an equivalence between the two systems. Are there solutions of equation (4) that cannot be represented as solutions of equation (24)? This is an important question because the generalized Taylor principle is useful only if uniqueness of a solution to the expanded linear system implies that equilibrium in the original model is also unique. The following section establishes that is not the case by presenting a counter example.

[3] Note that (78) and (79) in Appendix B of David and Leeper (2007, 631) are the same as equation (4) in this comment. By substituting (80) through (85) into (78) and (79), they rewrite the system in the expanded linear form given by (86) on page 632.

IV. A Counterexample to the Generalized Taylor Principle

The generalized Taylor principle is a statement about the set of solutions to the expanded linear model and, applied to this model, the statement is correct. While Davig and Leeper do not claim that their analysis applies to the original nonlinear Markov switching model, one might infer that this is the case.

The purpose of this section is to establish our claim that these two systems are not the same by constructing an example in which the generalized Taylor principle holds and hence the expanded linear model has a unique solution, but the original Markov switching system has a continuum of fundamental equilibria. Our example is based on the parameterization $\beta = 0.99$, $\sigma = 1.0$, and $\kappa = 0.17$. These parameters are taken from the baseline case, (David and Leeper 2007, 616). We choose $\gamma_1 = \gamma_2 = 0$, $p_{11} = 0.8$, and $p_{22} = 0.95$ as in the example used to construct Davig and Leeper's Figure 2 on page 617. We consider the two policy regimes represented by $\alpha_1 = 3.0$ and $\alpha_2 = 0.92$, where the parameter values in the second regime are the same as those used to construct an indeterminate example of the constant parameter model in Section II of this comment. Substituting our chosen values into the expressions for the matrices $\mathbf{A}$ and $\mathbf{B}$ from equations (25) and (26), one obtains the following values for the absolute values of the eigenvalues of $\mathbf{A}^{-1}\mathbf{B}$:

$$\begin{bmatrix} 1.5883 \\ 1.5883 \\ 1.2349 \\ 1.0167 \end{bmatrix}. \tag{27}$$

Since these all are greater than one, Davig and Leeper's generalized Taylor principle implies a unique bounded equilibrium of the expanded linear system, equation (24). But it does not imply that if this principle is satisfied the economic model, equation (4), has a unique equilibrium, as the following example demonstrates.

Using the parameter values introduced above, the matrix $\mathbf{\Gamma}_2$ from equation (4) has an eigenvalue of 0.9426 and an associated eigenvector of

$$\begin{bmatrix} -0.9306 \\ -0.3659 \end{bmatrix}. \tag{28}$$

Notice that this eigenvalue of 0.9426 is less than $p_{22} = 0.95$, and consider the following equations that represent our candidate fundamental equilibrium:

$$\mathbf{y}_t = \mathbf{G}_{s_t}\mathbf{u}_t + \mathbf{V}\mathbf{w}_t, \tag{29}$$

$$\mathbf{w}_t = \mathbf{\Lambda}_{s_t} w_{t-1} + M_{s_t,s_{t-1}} u_t, \tag{30}$$

where

$$\mathbf{V} = \begin{bmatrix} -0.9306 \\ -0.3659 \end{bmatrix}, \ \mathbf{\Lambda}_{s_t} = \begin{cases} 0 & \text{if } s_t = 1 \\ \dfrac{0.9426}{0.95} & \text{if } s_t = 2 \end{cases}, \ M_{s_t,s_{t-1}} = \begin{cases} 0 & \text{if } s_t = 1 \\ \mathbf{M}_{2,s_{t-1}}\mathbf{u}_t & \text{if } s_t = 2 \end{cases},$$

and $\mathbf{M}_{2,s_{t-1}}$ is any 1×2 real matrix that may or may not depend on s_{t-1}. Notice that $|\Lambda_i| < 1$ in both regimes and hence y_t is bounded. Importantly, $|\mathbf{\Lambda}_2|$ is equal to the smallest root of $\mathbf{\Gamma}_2$ divided by p_{22}, and the fact that it is less than one follows from our parameterization in which the smallest root of $\mathbf{\Gamma}_2$ equals 0.9426 and p_{22} is equal to 0.95. Notice that the form of our solution is close to those for the constant parameter case with the important difference that the parameter matrices $\mathbf{G}$ and $\mathbf{\Lambda}$ are different in different regimes. We chose to present a solution of this form because it is directly comparable to the form of solution we derive for the expanded linear system in Section V. In that section we discuss further the relationship between the two cases.

The following argument establishes that equations (29) and (30) do indeed define a solution to (4):

$$E_t y_{t+1} + \mathbf{\Psi}\mathbf{u}_t = p_{s_t,1} E_t[y_{t+1} | s_{t+1} = 1] + p_{s_t,2} E_t[y_{t+1} | s_{t+1} = 2] + \mathbf{\Psi}\mathbf{u}_t \tag{31}$$

$$= p_{s_t,1} \mathbf{V}\Lambda_1 w_t + p_{s_t,2} \mathbf{V}\Lambda_2 w_t + \mathbf{\Psi}\mathbf{u}_t \tag{32}$$

$$= p_{s_t,2} \mathbf{V}\mathbf{\Lambda}_2 \mathbf{w}_t + \mathbf{\Psi}\mathbf{u}_t \tag{33}$$

$$= \mathbf{\Gamma}_{s_t} \mathbf{V}\mathbf{w}_t + \mathbf{\Gamma}_{st} \mathbf{G}_{s_t} \mathbf{u}_t \tag{34}$$

$$= \mathbf{\Gamma}_{s_t} \mathbf{y}_t. \tag{35}$$

Equation (31) decomposes $E_t y_{t+1}$ using the law of iterated expectations. To obtain equation (32), substitute equations (29) and (30), led one period, into the right-hand side of equation (31) and take expectations, using the fact that u_{t+1} has zero mean and is independent of $\{s_{t+1}, s_t, \dots\}$. Equation (33) follows from the fact that $\Lambda_1 = 0$. To obtain equation (34), notice that if $s_t = 1$, then $p_{12}\mathbf{V}\Lambda_2\mathbf{w}_t = \mathbf{\Gamma}_1\mathbf{V}\mathbf{w}_t$ because $w_t = 0$, and if $s_t = 2$ then $p_{22}\mathbf{V}\Lambda_2\mathbf{w}_t = \mathbf{\Gamma}_2\mathbf{V}\mathbf{w}_t$ because $\mathbf{V}$ is the eigenvector of $\mathbf{\Gamma}_2$ corresponding to the eigenvalue $p_{22}\mathbf{\Lambda}_2$. This establishes that $p_{s_t,2}\mathbf{V}\Lambda_2\mathbf{w}_t = \mathbf{\Gamma}_{s_t}\mathbf{V}\mathbf{w}_t$. The second term of equation (34) follows from the construction of the fundamental solution, equation (21), $\mathbf{\Psi} = \mathbf{\Gamma}_{s_t}\mathbf{G}_{s_t}$. The final line follows from collecting terms in $\mathbf{\Gamma}_{s_t}$ and substituting for $\mathbf{y}_t$ from equation (29).

This argument establishes that equations (29) and (30) define a valid bounded solution to (4). More generally, such a construction can be performed whenever $\mathbf{\Gamma}_i$ has an eigenvalue less than p_{ii} in absolute value. In such cases, therefore, even if a central banker follows a policy that is consistent with Davig and Leeper's generalized Taylor principle, the economy may still be subject to indeterminacy.

V. Comparing Different Indeterminate Equilibria

Section IV demonstrates that the generalized Taylor principle rules out some, but not all, indeterminate fundamental equilibria. How do solutions ruled out by this principle differ from those that are not? In this section we show how to write indeterminate solutions to the regime-switching model included by Davig and Leeper's approach, and we compare them to those fundamental equilibria presented in Section IV.

If we assume that both $\mathbf{A}$ and $\mathbf{B}$, as given by equations (25) and (26), are invertible, and if there are $k \geq 1$ eigenvalues of $\mathbf{\Gamma} = \mathbf{A}^{-1}\mathbf{B}$ that are less than one in absolute value, we may use the analysis of Section II to write bounded indeterminate solutions to equation (24) in the form

$$Y_t = \mathbf{B}^{-1}\mathbf{C}u_t + \mathbf{V}\mathbf{w}_t, \tag{36}$$

$$\mathbf{w}_t = \mathbf{\Lambda} w_{t-1} + \mathbf{M}_{s_t, s_{t-1}} \mathbf{u}_t, \tag{37}$$

where (recall that) $\mathbf{Y}_t = (\pi_{1,t}, \pi_{2,t}, x_{1,t}, x_{2,t})'$, $\mathbf{u}_t = (u_t^S, u_t^D)'$, and the term $\mathbf{B}^{-1}\mathbf{C}$ is given by

$$\mathbf{B}^{-1}\mathbf{C} = \begin{bmatrix} \dfrac{\sigma+\gamma_1}{\sigma+\gamma_1+\kappa\alpha_1} & \dfrac{\kappa\sigma}{\sigma+\gamma_1+\kappa\alpha_1} \\ \dfrac{\sigma+\gamma_2}{\sigma+\gamma_2+\kappa\alpha_2} & \dfrac{\kappa\sigma}{\sigma+\gamma_2+\kappa\alpha_2} \\ \dfrac{-\alpha_1}{\sigma+\gamma_1+\kappa\alpha_1} & \dfrac{\sigma}{\sigma+\gamma_1+\kappa\alpha_1} \\ \dfrac{-\alpha_2}{\sigma+\gamma_2+\kappa\alpha_2} & \dfrac{\sigma}{\sigma+\gamma_2+\kappa\alpha_2} \end{bmatrix}, \tag{38}$$

where $\mathbf{V}$ is a $4 \times k$ matrix, $\mathbf{\Lambda}$ is a $k \times k$ matrix, and all the eigenvalues of $\mathbf{\Lambda}$ are less than one in absolute value. The matrices $\mathbf{V}$ and $\mathbf{\Lambda}$ satisfy the restriction $\mathbf{V\Lambda} = \mathbf{\Gamma V}$ and are obtained from the Schur decomposition of $\mathbf{\Gamma}$. Note that the fundamental shocks come from $\{s_t, s_{t-1}, \ldots, u_t, u_{t-1}, \ldots\}$, and thus the solution defined by equations (36) and (37) depends only on fundamentals.

The term $\mathbf{M}_{s_t, s_{t-1}}$ represents any $k \times 2$ real matrix that may depend on both the current and past regimes. Because $\mathbf{u}_t$ is mean zero and independent of $\{s_t, s_{t-1}, \ldots\}$, $E_{t-1}(\mathbf{M}_{s_t,s_{t-1}} u_t) = 0$, and it is straightforward to verify that equations (36) and (37) are bounded solutions to the expanded linear system (24). The proof is the same as that given for the constant parameter case in the paragraph following equation (12).

We have shown how to construct indeterminate solutions to Davig and Leeper's expanded linear system; but this leaves open the question: can one use bounded indeterminate solutions to equation (24) to construct solutions to the original nonlinear system? To answer this question, define the matrices $\mathbf{\Pi}_1$, and $\mathbf{\Pi}_2$ as follows:

$$\mathbf{\Pi}_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \text{ and } \mathbf{\Pi}_2 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \tag{39}$$

and let

$$\mathbf{V}_{s_t} = \mathbf{\Pi}_{s_t}\mathbf{V}. \tag{40}$$

Now premultiply equation (36) by $\mathbf{\Pi}_{s_t}$ and write it as two separate subsystems for $s_t = \{1,2\}$,

$$y_t = \mathbf{G}_{s_t}\mathbf{u}_t + \mathbf{V}_{s_t}\mathbf{w}_t, \text{ and} \tag{41}$$

$$w_t = \mathbf{\Lambda w}_{t-1} + \mathbf{M}_{s_t,s_{t-1}}\mathbf{u}_t, \tag{42}$$

where it follows from equations (6), (22), and (38) that

$$\mathbf{G}_{s_t} = \mathbf{\Gamma}_{s_t}^{-1}\mathbf{\Psi} = \mathbf{\Pi}_{s_t}\mathbf{B}^{-1}\mathbf{C}. \tag{43}$$

Notice that these equations are similar to those we presented in our counter example, with the exception that the parameter matrix, $\mathbf{\Lambda}$, is the same in both regimes. This restriction is key to understanding the difference between Davig and Leeper's indeterminate solutions, which are ruled out by the generalized Taylor principle, and solutions such as those presented in our counter example, which are not.

A comparison of equations (29) and (30) with (41) and (42) establishes that these two kinds of solutions have the first term in common, represented by $\mathbf{G}_{s_t}\mathbf{u}_t$, but differ in the second term since equation (29) restricts the matrix $\mathbf{V}_{s_t}$ to be the same across regimes, while equation (42) restricts $\boldsymbol{\Lambda}_{s_t}$ to be the same across regimes. Both solutions allow y_t to depend on the current and past policy regimes, as well as on the current and past demand and supply shocks, and all of the equilibria we consider in this comment are driven purely by fundamentals.

The main difference between the solution given by equations (29) and (30) and those given by equations (41) and (42) is the nature of the persistence of the process w_t. In equations (41) and (42) the persistence, which is governed by $\boldsymbol{\Lambda}$, is independent of the regime, while in equations (29) and (30), it can vary across regimes. Indeed, in a related paper, Farmer et al. (2007), we show that there exist general forms of indeterminate equilibria that include as special cases both the solution represented by equations (29) and (30) and the solution represented by equations (41) and (42).

We have demonstrated that the Davig-Leeper generalized Taylor principle does not exclude all bounded fundamental solutions to the original economic model. While the Davig-Leeper solutions are interesting, we see no economic reason to prefer one subset of fundamental equilibria over another. The Davig-Leeper solutions restrict $\boldsymbol{\Lambda}$ to be independent of regime, whereas the complete class allows this matrix to be different in different regimes.

VI. Conclusion

In the conventional linear new Keynesian model, there may be bounded equilibria in addition to the minimal state variable equilibrium for some parameter configurations. These indeterminate equilibria are serially correlated and add additional volatility to the time paths of the interest rate, output, and inflation. Since the influential empirical papers of Richard Clarida, Jordi Galí and Mark Gertler (2000) and Lubik and Schorfheide (2004), such equilibria represent leading candidate explanations for US time series data in the period before the Volcker disinflation of 1979–1982.

The advent of Markov switching models makes it possible to describe the periods before and after 1980 as a single rational expectations model and it leads to the question: were indeterminate equilibria responsible for the persistence and volatility observed in the postwar US time series data before 1980 after one accounts for the possibility that agents rationally anticipated the possibility of future regime change?

To answer this question, one would need to partition the parameter space into two subsets: one associated with indeterminacy and the other with a unique equilibrium. The parameter space in this more complete model includes not only the private sector and policy parameters in each regime, but also the probabilities of switching. Davig and Leeper's generalized Taylor principle takes a step toward answering this question. A complete partition of the parameter space into determinate and indeterminate regions remains an important but challenging open question.

REFERENCES

Clarida, Richard, Jordi Galí, and Mark Gertler. 2000. "Monetary Policy Rules and Macroeconomic Stability: Evidence and Some Theory." *Quarterly Journal of Economics*, 115(1): 147–80.

Davig, Troy, and Eric M. Leeper. 2007. "Generalizing the Taylor Principle." *American Economic Review*, 97(3): 607–35.

Farmer, Roger E. A., Daniel F. Waggoner, and Tao Zha. 2007. "Understanding the New-Keynesian Model when Monetary Policy Switches Regimes." National Bureau of Economic Research Working Paper 12965.

Lubik, Thomas A., and Frank Schorfheide. 2003. "Computing Sunspot Equilibria in Linear Rational Expectations Models." *Journal of Economic Dynamics and Control*, 28(2): 273–85.

Lubik, Thomas A., and Frank Schorfheide. 2004. "Testing for Indeterminacy: An Application to U.S. Monetary Policy." *American Economic Review*, 94(1): 190–217.

McCallum, Bennett T. 1983. "On Non-Uniqueness in Rational Expectations Models: An Attempt at Perspective." *Journal of Monetary Economics*, 11(2): 139–68.

Sims, Christopher A., and Tao Zha. 2006. "Were There Regime Switches in U.S. Monetary Policy?" *American Economic Review*, 96(1): 54–81.

Woodford, Michael. 2003. *Interest and Prices: Foundations of a Theory of Monetary Policy*. Princeton, NJ: Princeton University Press.

This article has been cited by:

1. William A. Branch, Troy Davig, Bruce McGough. 2012. ADAPTIVE LEARNING IN REGIME-SWITCHING MODELS. *Macroeconomic Dynamics* 1-25. [CrossRef]
2. Gianni Amisano, Oreste Tristani. 2011. Exact likelihood computation for nonlinear DSGE models with heteroskedastic innovations. *Journal of Economic Dynamics and Control* . [CrossRef]
3. Andrew P. Blake, Fabrizio Zampolli. 2011. Optimal policy in Markov-switching rational expectations models. *Journal of Economic Dynamics and Control* . [CrossRef]