Available online at www.sciencedirect.com

ScienceDirect

Journal of Economic Theory 144 (2009) 1849–1867

JOURNAL OF Economic Theory

www.elsevier.com/locate/jet

Understanding Markov-switching rational expectations models ☆

Roger E.A. Farmer [a], Daniel F. Waggoner [b], Tao Zha [b,c,*]

[a] *UCLA, United States*
[b] *Federal Reserve Bank of Atlanta, United States*
[c] *Emory University, United States*

Received 4 March 2008; final version received 7 May 2009; accepted 30 May 2009

Available online 4 July 2009

Abstract

We develop a set of necessary and sufficient conditions for equilibria to be determinate in a class of forward-looking Markov-switching rational expectations models and we develop an algorithm to check these conditions in practice. We use three examples, based on the new-Keynesian model of monetary policy, to illustrate our technique. Our work connects applied econometric models of Markov-switching with forward looking rational expectations models and allows an applied researcher to construct the likelihood function for models in this class over a parameter space that includes a determinate region and an indeterminate region.

JEL classification: C02; E40; E52

Keywords: Stability; Non-linearity; Unique equilibrium; Cross-regime indeterminacy; Expectations formation; Necessary and sufficient conditions

☆ This paper is a thorough revision of the earlier draft entitled "Understanding the new-Keynesian model when monetary policy switches regimes" (NBER, Working Paper 12965). We thank the referees and editors for thoughtful comments and Zheng Liu, Richard Rogerson, Eric Swanson, and John Williams for helpful discussions. We are grateful to Jacob Smith for excellent research assistance. This study is supported in part by NSF grant #0720839. The views expressed herein do not necessarily reflect those of the Federal Reserve Bank of Atlanta nor those of the Federal Reserve System.
* Corresponding author at: Research Division, Federal Reserve Bank of Atlanta, 1000 Peachtree Street, N.E., Atlanta, GA 30309-4470, United States. Fax: +1 404 498 8956.
E-mail address: tzha@earthlink.net (T. Zha).

0022-0531/$ – see front matter
doi:10.1016/j.jet.2009.05.004

1. Introduction

Reduced form Markov-switching models have been widely used to study economic problems in which there are occasional structural shifts in fundamentals. In an approach initiated by Hamilton [12], a set of economic time series is modeled as a vector autoregression (VAR) in which the parameters of the process are viewed as the outcome of a discrete state Markov process. It is well known that a constant parameter vector autoregression can be viewed as the reduced form of a forward looking rational expectations model but less is known about the Markov-switching case.

In a recent literature a number of authors have begun to study the relationship between Markov-switching models and forward looking Markov-switching rational expectations (MSRE) models. Work in this area includes papers by Leeper and Zha [14], Svensson and Williams [20], Blake and Zampolli [1], Davig and Leeper [7,8], and Farmer et al. [10]. MSRE models are more complicated than linear rational expectations models since the agents of the model must be allowed to take account of the possibility of future regime changes when forming expectations.

To make progress with empirical work that uses the MSRE approach one must be able to write down the likelihood function for a complete class of possible solutions. In the case of linear models, Lubik and Schorfheide [16] have shown how to partition the parameter space into two disjoint regions: one in which there exists a unique determinate rational expectations equilibrium and one in which there exist multiple indeterminate solutions driven by non-fundamental shocks. One would like to be able to find a similar partition for the case of MSRE models but, in order to accomplish this task, one would need to find a set of necessary and sufficient conditions under which an MSRE model has a unique determinate solution. This paper provides such conditions for an important subset of MSRE models; those in which there are no predetermined variables.

Our paper is structured in the following way. In Section 2 we discuss the relationship of our paper to previous literature. Section 3 introduces the class of forward looking Markov-switching rational expectations models that we will study and Section 4 reviews known results for the linear model. In Section 5 we discuss some results from the engineering literature and explain the differences between alternative stability concepts that are equivalent in linear models but different in Markov-switching systems. Sections 6 and 7 contain our main results; a characterization theorem and a set of necessary and sufficient conditions for determinacy of equilibrium. In Section 8 we provide an algorithm that is straightforward to apply in practice and in Section 9 we apply our results to a familiar example; that of the new-Keynesian model of monetary policy. Section 10 presents some concluding comments.

2. Related literature

Markov switching models in economics were first discussed by Hamilton [12], who applied them to autoregressive models of gdp where the parameters of the model are allowed to switch between two regimes. Forward-looking regime switching models have been studied by Svensson and Williams [20], Davig and Leeper [7,8] and Farmer et al. [9–11], who use them to study the effectiveness of monetary policy. We briefly review the issues that arise in that literature to explain why the current paper has relevance to an important body of applied research and the debate over the causes of an observed reduction in the volatility of macroeconomic variables in the period after 1980 – a phenomenon widely referred to as the Great Moderation.

In the context of this debate, Sims and Zha [19] use a backward-looking Markov-switching model to ask: Were there regime changes in US monetary policy? Their preferred explanation for the Great Moderation is that it was caused by changes to the shock variances of an identified vector autoregression. An alternative explanation, due to Cogley and Sargent [3,4], argues that changes in observed behavior of US time series is due to parameter drift in a random coefficient model.

Clarida et al. [2] and Lubik and Schorfheide [17] have presented a third view. They argue that the policy followed by the Fed before 1980 led to indeterminate equilibria that permitted non-fundamental 'sunspot' shocks to add volatility to realized outcomes. Although this explanation for the Great Moderation is intriguing, it is inconsistent with the rational expectations assumption: If policy has switched in the past, it might be expected to switch again in the future. Agents in the model studied by these authors do not take account of this possibility.

The papers of Svensson and Williams [20], Davig and Leeper [7,8] and Farmer et al. [9–11] extend the class of models studied by Clarida et al. and Lubik and Schorfheide to the Markov-switching rational expectations environment. This extension is important because it connects the reduced form econometric literature with structural economic theory and allows investigators to account for anticipation effects. In this environment it becomes possible to ask the question: Was the Great Moderation caused by a change in the parameters of the policy rule in a structural model or by a reduction in the variance of structural disturbances?

Although the MSRE literature has made some headway in addressing questions like this there has been, until now, no known set of necessary and sufficient conditions to determine if the parameters of a Markov-switching rational expectations model lead to a determinate equilibrium. Davig and Leeper [8] show that some solutions to the MSRE model have a linear representation and they find conditions for the solution to this linear representation to be unique; but Farmer et al. [10] show that these conditions do not apply to the original Markov-switching rational expectations model.

In the current paper we provide a complete set of necessary and sufficient conditions for a large class of forward looking MSRE models to be determinate. Our results provide the necessary tools for applied researchers to estimate structural models in this class using maximum likelihood methods.

3. The class of models

We study a class of ergodic multivariate forward-looking rational expectations models in which the parameters follow a discrete state Markov chain indexed by s_t with transition matrix $P = [p_{ij}]$. The element p_{ij} represents the probability that $s_t = j$ given $s_{t-1} = i$ for $i, j \in \{1, \ldots, h\}$ where $h \geqslant 1$ is the number of regimes and when $s_t = i$ we say that the system is in *regime* i.[1] The models we study are represented by the equation,

$$\Gamma_{s_t} y_t = E_t y_{t+1} + \Psi_{s_t} u_t, \tag{1}$$

where y_t is an n-dimensional vector of endogenous random variables with finite first and second moments, Γ_{s_t} is an invertible $n \times n$ matrix, Ψ_{s_t} is an $n \times m$ matrix, and u_t is an m-dimensional vector of exogenous shocks that are assumed to be stationary. While the existence of a solution to Eq. (1) depends on the properties of u_t, its uniqueness does not. Thus, to simplify the exposition, we assume without loss of generality that u_t is iid, mean-zero, and independent of the Markov process s_t.

[1] The engineering literature (Costa et al. [6]) uses *mode* to refer to what we call a regime.

We interpret y_t to be a vector of economic variables that depends on expectations of its own future value and we seek a solution to Eq. (1) that satisfies a suitable stability concept.

4. The linear case

To explain our approach, we will spend some time discussing the familiar case when $h = 1$ for which Eq. (1) is linear and can be written as follows,

$$\Gamma y_t = E_t y_{t+1} + \Psi u_t. \tag{2}$$

In this case a *solution* is a stable stochastic process that satisfies Eq. (2). Depending on the values of the parameters there may be one or more solutions.

One solution, referred to as a minimal state variable (MSV) solution following McCallum [18], describes y_t as a linear function of the fundamental shocks $\{u_t\}$. For Eq. (2), a solution of this kind exists and is given by the expression,

$$y_t = G u_t, \tag{3}$$

where

$$G = \Gamma^{-1}\Psi. \tag{4}$$

We require a solution to Eq. (2) to be stable because economic agents are assumed to base decisions on expectations of the future values of y_t and these expectations are obtained by recursively iterating Eq. (2) into the future; stability ensures that this process is well defined.

For some parameter configurations, and some definitions of stability, there may be an infinite set of solutions to Eq. (2) all of which are stable. When this occurs, each member of the set is said to be an indeterminate equilibrium. The minimal state variable solution is a member of this set but there may be other solutions that are serially correlated and add additional volatility to the time paths of the state variables.

In two recent papers on the empirical importance of indeterminate equilibria, Lubik and Schorfheide [16,17] show how to write an indeterminate solution as a linear combination of the minimal state variable solution and a first order moving average component. These solutions can be written as follows,

$$y_t = G u_t + w_t, \tag{5}$$

$$w_t = \Lambda w_{t-1} + V\gamma_t. \tag{6}$$

In these expressions, γ_t is a stable, k-dimensional, zero-mean, non-fundamental disturbance that may or may not be correlated with the fundamental shock u_t, k is the number of eigenvalues of Γ that are inside the unit circle and Λ is an $n \times n$ matrix of rank k, of the form

$$\Lambda = V\Phi V', \tag{7}$$

with

$$r_\sigma(\Phi) < 1. \tag{8}$$

The notation $r_\sigma(\Phi)$ denotes the spectral radius of Φ, which is the maximum of the absolute value of the eigenvalues of Φ. The $n \times k$ matrix V has orthonormal columns and the $k \times k$ matrix Φ is block upper triangular and its eigenvalues are the stable eigenvalues of Γ. The matrices V and Φ must satisfy

$$\Gamma V = V\Phi. \tag{9}$$

Note that it is *not true* in general that $\Gamma = V\Phi V'$ since Φ contains only a subset of the eigenvalues of Γ. Both the matrices Φ and V can be easily obtained from the real Schur decomposition of Γ.

There are two important lessons to be learned from the linear model. First, by writing solutions in the form of Eqs. (5) and (6) it is possible to convert the question of whether there is a unique determinate solution to Eq. (2) into the related question of whether Eq. (6) is a stable stochastic process. Second, to answer the determinacy question we must settle on a suitable concept of stability.

In the following section we will define two concepts: mean-square stability and bounded stability. These concepts are equivalent in the linear model but, in models with Markov-switching, they are no longer the same. We will explain why engineers chose mean-square stability as the appropriate stability concept over bounded stability and discuss some lessons that can be learned from the engineers.

5. What engineering has to teach us

Our strategy for finding necessary and sufficient conditions for determinacy is to show that solutions to Eq. (1) have a similar representation to the moving average solutions, Eqs. (5) and (6), that solve the linear model. This turns the determinacy question into one of stability and allows us to appeal to theorems from the engineering literature on the existence and uniqueness of stable solutions to a class of equations that economists call Markov-switching models and engineers refer to as *discrete-time Markov jump linear systems*.[2] These are VARs in which the parameters are governed by a discrete state Markov chain and they can be represented by the following expression,

$$x_t = A_{s_t} x_{t-1} + B_{s_t} \xi_t, \tag{10}$$

where x_t is an n-dimensional stochastic process, A_{s_t} is an $n \times n$ matrix, B_{s_t} is an $n \times m$ matrix, and ξ_t is a stable m-dimensional process independent of the Markov process s_t.

Our main idea is to show that all solutions to Eq. (1) can be written as the sum of two particular solutions, one of which depends only on the current regime and the other is a Markov-switching system with the same form as Eq. (10). We are thus able to convert the question of whether Eq. (1) has a unique determinate solution to the equivalent question of whether Eq. (10) possesses a unique stable solution. This approach requires that we define what it means for the solution to a Markov-switching model to be stable.

The Markov-switching system described by Eq. (10) is mean-square stable if its first and second moments converge to well-defined limits as the horizon extends to infinity. If, in addition, the process is also bounded, then we say it is boundedly stable. The formal definitions of mean-square stability and bounded stability are given below.

Definition 1. An n-dimensional process x_t is mean-square stable (MSS) if and only if there exists an n-vector μ and an $n \times n$ matrix Σ such that

(a) $\lim_{t\to\infty} E_0[x_t] = \mu$,
(b) $\lim_{t\to\infty} E_0[x_t x_t'] = \Sigma$.

[2] Since there is a large economics literature that uses the term Markov-switching system, we will use the prevailing economic terminology from this point on.

Definition 2. An n-dimensional process x_t is bounded if there exists a real number N such that

$$\|x_t\| < N, \quad \text{for all } t,$$

where $\|\cdot\|$ is any well-defined norm. If, in addition, the process is MSS, then the process is said to be boundedly stable.

Other notions of stability could be used in place of mean-square stability. For instance, in economics covariance stationarity is often used, or asymptotic covariance stationarity can be used if one wishes to avoid taking a stand on initial conditions.[3] In general, asymptotic covariance stationarity is strictly stronger than mean-square stability. For the system given by Eq. (10), however, under the assumption that the innovation process ξ_t is asymptotically covariance stationary, the system will mean-square stable if and only if it is asymptotically covariance stationary. Because many of the standard theorems we use in this paper are stated in terms of mean-square stability, we also use mean-square stability, but the reader should note that the results of our paper would hold if asymptotic covariance stationarity were used instead. Throughout this paper, we shall use the terms stability and mean-square stability interchangeably.

For linear systems with bounded shocks, mean-square stability and bounded stability are equivalent concepts for determining uniqueness of the equilibrium. For Markov-switching models, however, these two concepts *are not the same* and one must choose between them. Engineers use mean-square stability for several reasons. First, there are many instances of engineering problems in which the system may be unstable in one or more of its regimes. But as long as this regime does not occur too frequently the state variables will still converge to a well-defined ergodic distribution with finite first and second moments. Unstable regimes would be ruled out by bounded stability and this definition of stability would define many interesting physical phenomena to be unstable even though they possess well-behaved limiting distributions. Second, most practical applications assume that the system is driven by unbounded errors; for example, normal or lognormal distributions are frequently assumed, and hence the state variables are also unbounded in practical applications. Third, bounded stability is difficult to work with and there is no known set of necessary and sufficient conditions under which a Markov-switching system will display stability in this stronger sense.

All three of these issues arise in economics. Economics, like engineering, is ripe with examples where one or more regimes are unstable. For example, hyperinflation in Argentina in the 1980's and 1990's was characterized by a series of explosive regimes that were subsequently stabilized. Second, although economic theorists often use boundedness as a stability concept, applied researchers typically use shock distributions with unbounded errors. Finally, to see why bounded stability poses a practical difficulty, consider the following example from Costa et al. [6, p. 39].

Consider a two-dimensional system

$$x_t = A_{s_t} x_{t-1}, \tag{11}$$

where A_i is a 2×2 matrix that take the following values,

$$A_1 = \begin{bmatrix} 0 & 2 \\ 0 & 0.5 \end{bmatrix} \quad \text{and} \quad A_2 = \begin{bmatrix} 0.5 & 0 \\ 2 & 0 \end{bmatrix}, \tag{12}$$

[3] Some authors use the terms "wide-sense stationarity" and "asymptotic wide-sense stationarity" instead of "covariance stationarity" and "asymptotic covariance stationarity."

and consider the following two alternative transition matrices,

$$\begin{bmatrix} 0.8 & 0.2 \\ 0.4 & 0.6 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0.9 & 0.1 \\ 0.4 & 0.6 \end{bmatrix}. \tag{13}$$

In this example the roots of A_1 and A_2 both lie inside the unit circle and if the state were to remain in either regime 1 or regime 2 the system would be stable. Although both roots of A_1 and both roots of A_2 are inside the unit circle, the product $A_1 A_2$ has a root outside the unit circle. This means that as long as the system alternates between regime 1 and 2, the process x_t would grow exponentially and hence the system is unbounded *regardless* of the values of the transition matrix.

In general, for a process given by Eq. (11) to be bounded, all roots of all possible products of A_1 and A_2 must be inside the unit circle. For boundedness to be a workable stability concept one would need to find a simple condition under which all possible products of the coefficient matrices have roots inside the unit circle and to our knowledge, no such condition is known.

Although there is no known way to check for bounded stability, mean-square stability is much easier to deal with and Costa et al. [6, pp. 34–36] show that mean-square stability of (11) is equivalent to the question: Are all roots of the matrix

$$\begin{bmatrix} p_{1,1} A_1 \otimes A_1 & p_{2,1} A_2 \otimes A_2 \\ p_{1,2} A_1 \otimes A_1 & p_{2,2} A_2 \otimes A_2 \end{bmatrix} \tag{14}$$

inside the unit circle? For the first transition matrix given in (13), the matrix expressed in (14) has a root outside the unit circle and so the system is unstable even though it is stable in each regime separately, a somewhat surprising result. On the other hand, for the second transition matrix given in (13), all the roots of the matrix expressed in (14) are inside the unit circle and hence the system is stable. These results illustrate that mean-square stability and bounded stability are different stability concepts.

6. A definition and a characterization theorem

In this section we move beyond the engineering literature. Our main result is to show that the complete set of solutions to the MSRE model, Eq. (1), can be described as the sum of two particular solutions. One is what McCallum [18] has called a *minimal state variable solution* (MSV) and the other is a Markov-switching system.

We begin by defining a rational expectations equilibrium and proving a theorem that characterizes a class of stochastic processes that satisfy this definition.

Definition 3. A rational expectations equilibrium is a mean-square stable stochastic process that satisfies Eq. (1).

The following theorem characterizes all possible solutions to Eq. (1), whether or not they satisfy mean-square stability.[4]

Theorem 1. *Any solution to the MSRE model* (1) *can be written in the following way*:

[4] This includes solutions from our previous work Farmer et al. [10,11] which, superficially, appear to have a different form.

$$y_t = G_{s_t} u_t + w_t, \tag{15}$$
$$w_t = \Lambda_{s_{t-1},s_t} w_{t-1} + V_{s_t} V_{s_t}' \gamma_t, \tag{16}$$

where V_{s_t} is an $n \times k_{s_t}$ matrix with orthonormal columns and $0 \leqslant k_{s_t} \leqslant n$, γ_t is an arbitrary n-dimensional shock process such that $E_{t-1}[V_{s_t} V_{s_t}' \gamma_t] = 0$, Λ_{s_{t-1},s_t} is an $n \times n$ matrix of the form $V_{s_t} \Phi_{s_{t-1},s_t} V_{s_{t-1}}'$ for some $k_{s_t} \times k_{s_{t-1}}$ matrix Φ_{s_{t-1},s_t} such that

$$\Gamma_i V_i = \sum_{j=1}^{h} p_{i,j} V_j \Phi_{i,j} \quad \text{for } 1 \leqslant i \leqslant h, \tag{17}$$

and $G_{s_t} u_t$ is the minimum-state-variable (MSV) solution with $G_{s_t} = \Gamma_{s_t}^{-1} \Psi_{s_t}$.

Proof. See Appendix A. □

This theorem states that all solutions to the MSRE model can be written in an analogous form to the Lubik–Schorfheide representation of solutions to the linear system; recall that these were represented as the sum of a minimal state variable solution and a moving average component. Our contribution is to show that, in the case of MSRE models, the moving average component is a Markov-switching system and it is this theorem that allows us to appeal to results from the engineering literature to find conditions for the model defined by Eq. (1) to be determinate.

The most important part of our result is Eq. (17), which plays the role of Eq. (9) for the linear system. Whereas the matrices Φ and V in Eq. (9) can be obtained directly from the real Schur decomposition of Γ, techniques to find the set of matrices $\Phi_{i,j}$ and V_i are more involved. As we will see in the next section, Eq. (17) will be key in devising an algorithm for determining whether or not there are multiple stable solutions to Eq. (1).

7. Necessary and sufficient conditions for a unique equilibrium

In this section we develop a set of necessary and sufficient conditions for the existence of a unique rational expectations equilibrium. We assume that the sunspot random process γ_t has mean zero, is mean-square stable, and is independent of the fundamental Markov process s_t. We introduce the following definition:

$$M_1(X_{i,j}) = \begin{bmatrix} p_{1,1} X_{1,1} \otimes X_{1,1} & \cdots & p_{h,1} X_{h,1} \otimes X_{h,1} \\ \vdots & \ddots & \vdots \\ p_{1,h} X_{1,h} \otimes X_{1,h} & \cdots & p_{h,h} X_{h,h} \otimes X_{h,h} \end{bmatrix}. \tag{18}$$

The matrices $M_1(\Lambda_{i,j})$ and $M_1(\Phi_{i,j})$ are important because they play a role in expressing the variance of w_t in terms of the variance of w_{t-1}. The details of this are outlined in the proof of the following theorem.

Theorem 2. *The process represented by Eqs.* (15)–(17) *is a mean-square stable solution to the MSRE model* (1) *if and only if*

$$r_\sigma\big(M_1(\Phi_{i,j})\big) < 1. \tag{19}$$

Proof. See Appendix A. □

Since a rational expectation equilibrium is defined as a mean-square stable solution to Eq. (1), Theorem 2 provides necessary and sufficient conditions for determinacy; that is, for the rational expectations equilibrium to be unique. In the linear case the analogous conditions can easily be checked using the real Schur decomposition of a known matrix. For the Markov-switching case, however, the conditions provided in Theorem 2 are difficult to verify in practice because finding V_i and $\Phi_{i,j}$ that satisfy Eqs. (15)–(17) is not a standard problem in matrix algebra. It is, however, equivalent to a collection of constrained optimization problems.

For each choice of dimensions $\{k_1, \ldots, k_h\}$, Eq. (17) and the orthogonality conditions on the columns of V_i provide constraints on the V_i and $\Phi_{i,j}$. Subject to these constraints, one minimizes the objective function $r_\sigma(M_1(\Phi_{i,j}))$. If the minimum value of the objective function is less than one, then the V_i and $\Phi_{i,j}$ that give the minimun define an MSS solution different from the MSV solution. On the other hand, if for all choices of dimensions $\{k_1, \ldots, k_h\}$, not all zero, the minimum value is greater than or equal to one, then there can be no MSS solution other than the MSV solution. This is formalized in the following corollary.

Corollary 1. *Let $0 \leqslant k_i \leqslant n$. Consider the problem of choosing $n \times k_i$ matrices V_i and $k_j \times k_i$ matrices $\Phi_{i,j}$ such that $r_\sigma(M_1(\Phi_{i,j}))$ is minimized subject to the constraints $\Gamma_i V_i = \sum_{j=1}^{h} p_{i,j} V_j \Phi_{i,j}$ and $V_i' V_i = I_{k_i}$.*

(1) *If there exists some choice of $\{k_1, \ldots, k_h\}$, not all zero, such that the optimal solution $r_\sigma(M_1(\Phi_{i,j}))$ is smaller than one, there will be multiple solutions to Eq.* (1).
(2) *If, for all possible choices of $\{k_1, \ldots, k_h\}$, not all zero, the minimum value of $r_\sigma(M_1(\Phi_{i,j}))$ is greater than or equal to one, there will be only one mean-square stable solution to the MSRE model* (1). *This solution is the MSV solution.*

Proof. The proof follows directly from Theorem 2. □

8. Constructing sunspot solutions

While Theorem 2 and its corollary provide a general technique for determining if there is a unique MSS solution of Eq. (1), we consider a special case where we are able to explicitly solve the constraints and thus characterize this class of solutions. Given the complexity of the problem, this illustrative case is important because it allows one to develop intuition about MSS solutions other than the MSV solution and relate our solution technique to the standard eigenvalue problem. It is also a case with practical relevance as we demonstrate in Section 9.

The case we consider is that of $k_i = 1$ for $1 \leqslant i \leqslant h$. This implies that V_i will be a vector and $\Phi_{i,j}$ will be a scalar. The following proposition characterizes solutions of this form.

Proposition 1.

(1) *If there exist scalars $c_1, \ldots, c_h$ and an nh-dimensional vector $v = (v_1, \ldots, v_h)$ with $v_i \neq 0$ such that*

$$\big(\operatorname{diag}(\Gamma_i) - \big(\big(\operatorname{diag}(c_i)P\big) \otimes I_n\big)\big)v = 0, \tag{20}$$

$$r_\sigma\big(\operatorname{diag}\big(c_i^2\big)P\big) < 1, \tag{21}$$

then there exist mean-square stable solutions to Eq. (1), *other than the MSV solution.*

(2) *These solutions are given by Eqs.* (15) *and* (16) *where*

$$V_{s_t} = \frac{v_{s_t}}{\|v_{s_t}\|}, \tag{22}$$

$$\Lambda_{s_{t-1},s_t} = \frac{c_{s_{t-1}}}{\|v_{s_{t-1}}\|^2} v_{s_t} v'_{s_{t-1}}. \tag{23}$$

(3) *The solutions defined in part* (2) *are bounded if and only if* $|c_i| < 1$.

Proof. See Appendix A. □

Part (3) of this proposition gives a complete characterization of the set of *boundedly stable* solutions of this form. Recall that in general one would need to check if all possible permutations of all possible products of matrices have roots inside the unit circle. This is simple in the case of $k_i = 1$ since the relevant matrices become scalars and scalar multiplication is commutative.

The proposition allows us to check for mean-square stable solutions by solving the non-linear equation

$$\det\bigl(\operatorname{diag}(\Gamma_i) - \bigl(\bigl(\operatorname{diag}(c_i)P\bigr) \otimes I_n\bigr)\bigr) = 0, \tag{24}$$

and seeing if Eq. (21) holds. In the case of one regime, this optimization problem reduces to checking to see if Γ has an eigenvalue inside the unit circle.

In the case of two or more regimes the numbers c_i are analogous to eigenvalues, and are eigenvalues if all the c_i are forced to be equal. However, in general, there will be a continuum of solutions of Eq. (24). For example, when $h = 2$, Eq. (24) defines a pair of curves $c_1 = \psi(c_2)$, which correspond to the two branches of the solution. The question of determinacy amounts to asking whether the correspondence $c_1 = \psi(c_2)$ has an intersection with the region defined by the spectral radius condition, Eq. (21). In Section 9, we provide economic examples and plot the correspondence $c_1 = \psi(c_2)$ for each example. We provide one example where there is a unique solution and two examples where there is a continuum of solutions for which the spectral radius condition is satisfied.

9. An application to the new-Keynesian model

In this section, we apply our theoretical results to the canonical new-Keynesian model studied by Lubik and Schorfheide [17][5];

$$\text{AS curve} \quad \pi_t = \beta E_t \pi_{t+1} + \kappa x_t + u_t^S, \tag{25}$$

$$\text{IS curve} \quad x_t = E_t x_{t+1} - \sigma^{-1}(i_t - E_t \pi_{t+1}) + u_t^D, \tag{26}$$

$$\text{Policy rule} \quad i_t = \alpha_{s_t} \pi_t + \iota_{s_t} x_t, \tag{27}$$

where x_t is the deviation of output from its trend path, π_t is a percentage deviation from its steady state value, i_t is the nominal interest rate, u_t^D is an aggregate demand shock, u_t^S is an aggregate supply shock and we allow for both u_t^D and u_t^S to be serially correlated:

$$u_t^S = \rho_S u_{t-1}^S + \varepsilon_t^S, \tag{28}$$

$$u_t^D = \rho_D u_{t-1}^D + \varepsilon_t^D. \tag{29}$$

[5] Liu et al. [15] show how to derive this MSRE model directly from the consumers and firms' optimization problems.

The innovations ε_t^S and ε_t^D are stationary exogenous random processes satisfying $E_t\varepsilon_{t+1}^S = E_t\varepsilon_{t+1}^D = 0$.

The private sector block, consisting of Eqs. (25) and (26), has three regime-independent parameters, σ, β and κ. The parameter σ represents the intertemporal elasticity of substitution, β is the discount factor of the representative household, and κ is the slope of the Phillips curve. Uncertain monetary policy, represented by Eq. (27), has two regime-dependent parameters, α_{s_t} and ι_{s_t}, that capture the degree to which monetary policy is active or passive and we concentrate on the case of two regimes by setting $h = 2$.

To write the new-Keynesian model in compact form, we substitute Eq. (27) into Eq. (26). Rearranging terms the model can then be written as

$$F_{s_t} y_t = H E_t y_{t+1} + u_t, \tag{30}$$

where

$$y_t = \begin{bmatrix} \pi_t \\ x_t \end{bmatrix}, \qquad u_t = \begin{bmatrix} u_t^S \\ u_t^D \end{bmatrix},$$
$$F_{s_t} = \begin{bmatrix} 1 & -\kappa \\ \sigma^{-1}\alpha_{s_t} & 1+\sigma^{-1}\iota_{s_t} \end{bmatrix}, \qquad H = \begin{bmatrix} \beta & 0 \\ \sigma^{-1} & 1 \end{bmatrix}.$$

This is a special case of Eq. (1) where $\Gamma_{s_t} = H^{-1}F_{s_t}$ and $\Psi_{s_t} = H^{-1}$.

The solution to this new-Keynesian model has the same form as Eqs. (15) and (16) but because we allow for autocorrelated errors, the coefficient matrices G_{s_t} (for $s_t = 1, \dots, h$) for the MSV solution have the following form:

$$\operatorname{vec}\begin{bmatrix} G_1 \\ \vdots \\ G_h \end{bmatrix} = \left[I_2 \otimes \operatorname{diag}(\Gamma_i) - \rho' \otimes P \otimes I_2\right]^{-1} \operatorname{vec}\begin{bmatrix} \Psi_1 \\ \vdots \\ \Psi_h \end{bmatrix}, \tag{31}$$

where ρ is a 2×2 diagonal matrix whose diagonal vector is $[\rho_S, \rho_D]$. For this model we can make use of Proposition 1 to find values of c_1 and c_2 such that $r_\sigma < 1$.

Following Leeper [13], the literature on Taylor Rules defines a regime in which the interest rate is changed by more than one for one in response to a change in expected inflation, to be an *active* regime. If the interest rate responds less than one for one, the regime is said to be passive. The response coefficient of the Fed is represented by the parameter α and the fact that regime 1 is passive and regime 2 is active is represented in our model by setting $|\alpha_1| < 1$ and $|\alpha_2| > 1$. The matrix P determines the persistence of each regime.

We will provide three examples, one example with a unique determinate equilibrium and two examples with a continuum of indeterminate equilibria and we will illustrate Proposition 1 in graphs. For all the examples we choose the following parameter values:

$$\alpha_1 = 0.8, \qquad \iota_1 = 0.0, \qquad \iota_2 = 0.0 \quad \text{(regime dependent parameters);}$$
$$\beta = 0.99, \qquad \sigma = 1.0, \qquad \kappa = 0.132, \qquad \rho_S = \rho_D = 0.9$$
$$\text{(private-sector parameters);}$$
$$p_{22} = 0.95 \quad \text{(probability of staying in the second regime).}$$

Whether a model is determinate or not is a complicated question and the answer depends on many parameter values in the model. In this section, we vary the values of only α_2 and p_{11} and focus on an illustration which demonstrates that uniqueness of the equilibrium can be affected

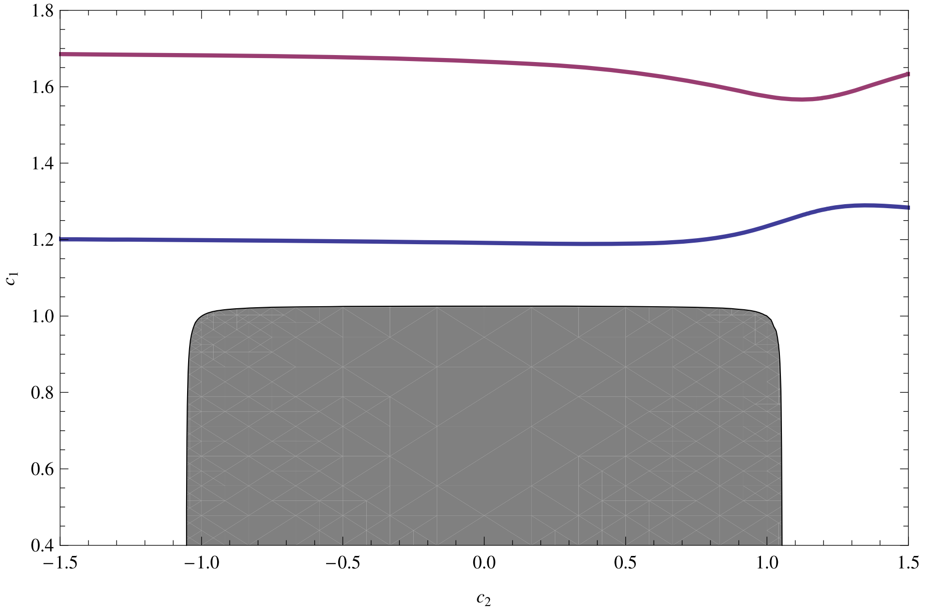

Fig. 1. The new-Keynesian model with the baseline parameterization: $\alpha_1 = 0.8$, $\alpha_2 = 2.5$, $\iota_1 = \iota_2 = 0.0$, $\beta = 0.99$, $\sigma = 1.0$, $\kappa = 0.132$, $p_{11} = 0.75$, and $p_{22} = 0.95$. The two lines are defined by Eq. (24) and the shaded region is defined Inequality (21).

by the degree of an active monetary policy and by the persistence of the active regime relative to the passive regime.

9.1. Example 1

In this example, we set

$$\alpha_2 = 2.5, \qquad p_{11} = 0.75.$$

For this parameterization, one can verify that part (2) in Corollary 1 is satisfied. Thus, the equilibrium is unique and characterized by the MSV solution, whose coefficient matrices are given by Eq. (31).

Fig. 1 gives an intuitive explanation of why the equilibrium is unique. Given the parameter values in this example, we compute the values $c_1 = \psi(c_2)$ for every value of c_2. The correspondence $c_1 = \psi(c_2)$ has two branches for a given value of c_2: we plot both branches in Fig. 1 as two lines both of which define values of c_1 and c_2 that satisfy Eq. (20). We also compute the region defined by Inequality (21). Fig. 1 shows that for this case there is an *empty* intersection of the correspondence with the region and hence there are no stable equilibria other than the MSV solution. All values of the correspondence $c_1 = \psi(c_2)$ represent solutions to Eq. (1) that are unstable and hence are ruled out by Definition 3.

If there were no switching between regimes, regime 1 would be associated with an indeterminate equilibrium and regime 2 would be associated with a unique determinate equilibrium. This follows from the fact that for our chosen parameters, the matrix Γ_1 has a root inside the unit circle whereas Γ_2 has both roots outside. When there is Markov switching between regimes, it is the *set* of regimes that is determinate.

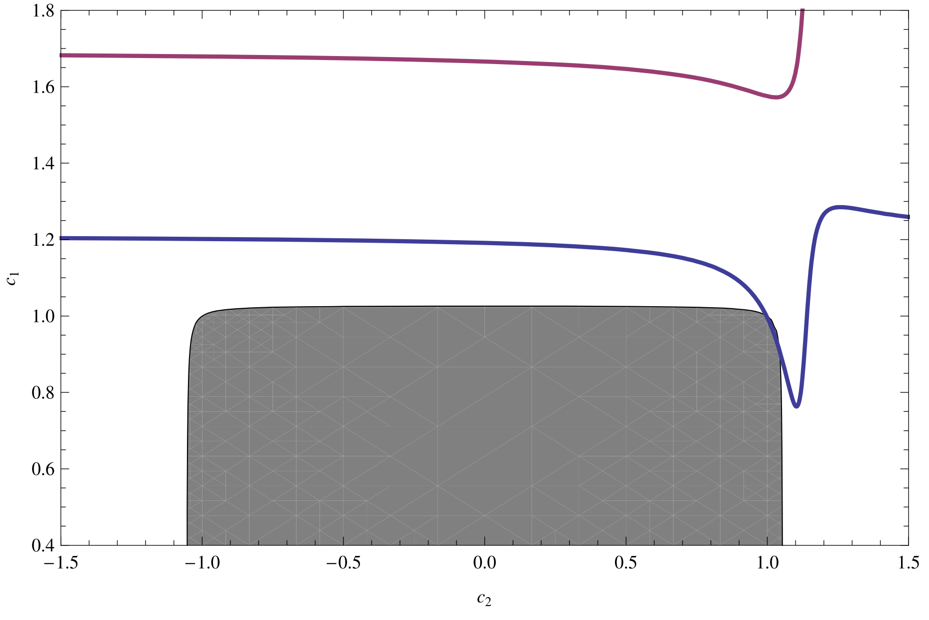

Fig. 2. The new-Keynesian model with an alternative parameterization: $\alpha_1 = 0.8$, $\alpha_2 = 1.05$, $\iota_1 = \iota_2 = 0.0$, $\beta = 0.99$, $\sigma = 1.0$, $\kappa = 0.132$, $p_{11} = 0.75$, and $p_{22} = 0.95$. The two lines are defined by Eq. (24) and the shaded region is defined Inequality (21).

9.2. *Example 2*

In this example, we set

$$\alpha_2 = 1.05, \qquad p_{11} = 0.75.$$

This example differs from the previous example only in that α_2 is smaller. Even with this smaller value of α_2, regime 2 by itself would be associated with a unique determinate equilibrium because Γ_2 has both roots outside the unit circle. We now use this example to demonstrate that the magnitude of an active policy's response to inflation plays an essential role in determining uniqueness of the equilibrium. The active policy in this case is too weak to ensure a unique equilibrium, as shown in Fig. 2. Recall that the correspondence $c_1 = \psi(c_2)$ shows values of c_1 and c_2 that satisfy Eq. (20) and shaded region represents points that satisfy Inequality (21). Using part (1) of Proposition 1; the fact that the lower branch of the correspondence intersects with the upper right corner shaded region implies that there is a continuum of indeterminate stable equilibria.

9.3. *Example 3*

A third example is provided by the following parameterization:

$$\alpha_2 = 1.05, \qquad p_{11} = 0.90.$$

Here, the duration of regime 1 relative to regime 2 is longer than that in Example 2. As discussed above, regime 1 in isolation would be associated with an indeterminate equilibrium. We saw in

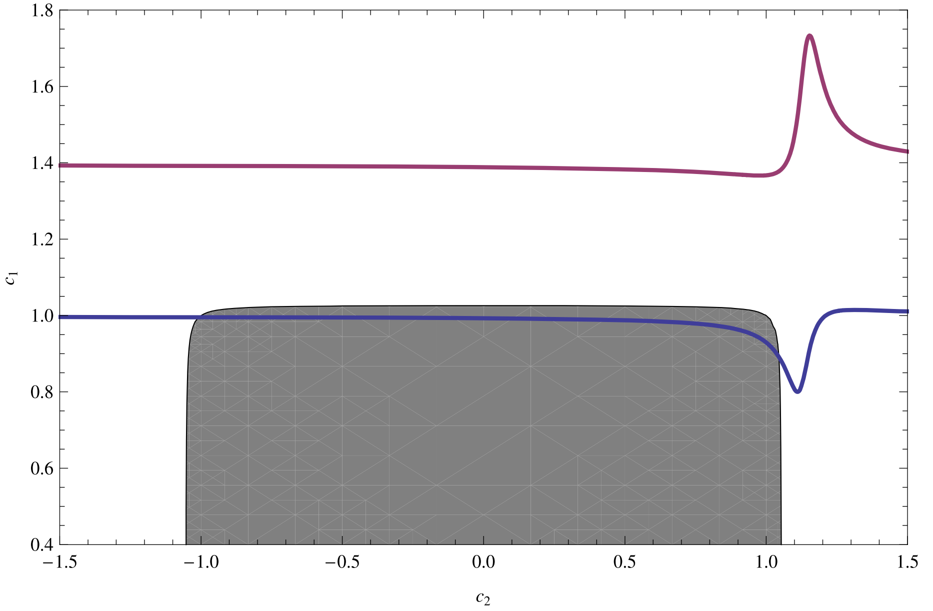

Fig. 3. The new-Keynesian model with an alternative parameterization: $\alpha_1 = 0.8$, $\alpha_2 = 1.05$, $\iota_1 = \iota_2 = 0.0$, $\beta = 0.99$, $\sigma = 1.0$, $\kappa = 0.132$, $p_{11} = 0.90$, and $p_{22} = 0.95$. The two lines are defined by Eq. (24) and the shaded region is defined Inequality (21).

Example 2 that there was only a small set of values of $c_1 = \psi(c_2)$ and c_2 that were associated with mean-square stable equilibria. Fig. 3 demonstrates that, when the passive regime is more persistent, the lower branch of the correspondence $c_1 = \psi(c_2)$ intersects with a much larger portion of the shaded region than in Example 2.

Example 3 is interesting because the characteristics of the indeterminate solutions display an unexpected property; the persistence of the propagation mechanism in the active regime can take a wide range of values. The parameters c_1 and c_2 determine the persistence of the propagation mechanism in each regime. Recall that the policy maker in regime 1 follows a passive policy and the policy maker in regime 2 is active.

From the scales on both the x-axis and the y-axis in Fig. 3 one can see that there is not much variation in the admissible values of c_1 while indeterminacy exists for a wide range of c_2. It follows from the fact that the range of c_1 inside the indeterminate region is small, that the characteristics of indeterminate dynamics in regime 1 are similar for the entire set of indeterminate equilibria.

There is, however, a wide range of possible values of c_2 that are consistent with an indeterminate equilibrium. Monetary policy is active in regime 2, and if regime 2 were an absorbing state, then once the system entered regime 2 the equilibrium would be determinate. But since the system can escape back to the passive regime, indeterminacy may spillover to the active regime and lead to many possible dynamic paths for the state variables in regime 2, *even though the Fed follows an active policy*. Further, the characteristics of each of these equilibria varies widely.

The parameter c_2 represents the degree of autocorrelation of the non-fundamental shock in the active regime and Fig. 3 shows that this can vary from -1 to a value greater then 1 with every possible value in between. Each of these values represents a different equilibrium with a very different degree of persistence for the observed behavior of inflation, output and the interest rate.

10. Conclusion

Our main contribution in this paper was to provide a set of necessary and sufficient conditions for determinacy in a class of forward looking Markov-switching rational expectations models. To accomplish this task, we showed how the question of determinacy of a rational expectations model can be restated as a stability question in a class of Markov-switching models. To make progress, we argued for the use of mean-square stability rather than bounded stability as the appropriate stability concept.

A second important contribution was to show how determinacy can be restated as a constrained optimization problem. This restatement permits an applied researcher to partition the parameter space of an economic model into determinate and indeterminate regions for a large class of Markov-switching rational expectations models and hence, to compute the likelihood for each regime.

Finally, we provided an application of our approach in the context of the familiar new-Keynesian model. For this example, the constrained optimization problem is amenable to a graphical analysis. We believe that our technique will provide useful in a wide variety of practical applications and we hope to extend it in future work to the case where the state vector may contain one or more predetermined variables.

Appendix A. Proofs

Proof of Theorem 1. Let y_t be a solution of Eq. (1). We must show that y_t can be represented by Eqs. (15)–(17). Define w_t by $w_t = y_t - G_{s_t} u_t$. By substituting this expression into Eq. (1) and making use of the definition

$$G_{s_t} = \Gamma_{s_t}^{-1} \Psi_{s_t}, \tag{A.1}$$

it follows that the process w_t is a solution of

$$\Gamma_{s_t} w_t = E_t[w_{t+1}]. \tag{A.2}$$

We must show that w_t can be expressed in the form given by Eqs. (16) and (17). Let V_i be any matrix with orthonormal columns such that the column space of V_i is the span of the support of $w_t 1_{\{s_t=i\}}$ for all t, where $1_{\{s_t=i\}}$ denotes the indicator function that is one if $s_t = i$ and zero otherwise.[6] Let k_i be the dimension of the column space of V_i. Since w_t is a solution of Eq. (A.2), the following equation holds almost surely:

$$\begin{aligned}\Gamma_i v &= E[\Gamma_{s_t} w_t \mid w_t = v,\ s_t = i] = E\big[E_t[w_{t+1}] \mid w_t = v,\ s_t = i\big] \\ &= E[w_{t+1} \mid w_t = v,\ s_t = i] = \sum_{j=1}^{h} p_{i,j} E[w_{t+1} \mid w_t = v,\ s_t = i,\ s_{t+1} = j].\end{aligned}$$

Because the column space of V_j contains the span of the support of $w_{t+1} 1_{\{s_{t+1}=j\}}$, it follows that $E[w_{t+1} \mid w_t = v,\ s_t = i,\ s_{t+1} = j]$ is almost surely in the column space of V_j. This and the fact

[6] If the support of $w_t 1_{\{s_t=i\}}$ is $\{0\}$, then we take V_i to be the $n \times 0$ matrix and follow the usual conventions of dealing with matrices that have a zero dimension.

that the column space of V_i is the span of the support of $w_t 1_{\{s_t=i\}}$ for all t, implies that there exists a $k_j \times k_i$ matrix $\Phi_{i,j}$ such that

$$\Gamma_i V_i = \sum_{j=1}^{h} p_{i,j} V_j \Phi_{i,j}.$$

Define $\gamma_t = w_t - V_{s_t} \Phi_{s_{t-1},s_t} V'_{s_{t-1}} w_{t-1}$. Because w_t, and hence γ_t, is almost surely in the column space of V_{s_t}, $\gamma_t = V_{s_t} V'_{s_t} \gamma_t$. All that remains to be shown is that $E_{t-1}[V_{s_t} V'_{s_t} \gamma_t] = 0$. Since

$$\begin{aligned} E_{t-1}\left[V_{s_t} V'_{s_t} \gamma_t\right] &= E_{t-1}\left[w_t - V_{s_t} \Phi_{s_{t-1},s_t} V'_{s_{t-1}} w_{t-1}\right] \\ &= \Gamma_{s_{t-1}} w_{t-1} - \sum_{j=1}^{h} p_{s_t,j} V_j \Phi_{s_{t-1},j} V'_{s_{t-1}} w_{t-1} \\ &= \Gamma_{s_{t-1}} w_{t-1} - \Gamma_{s_{t-1}} V_{s_{t-1}} V'_{s_{t-1}} w_{t-1} \\ &= 0, \end{aligned}$$

where the last equality holds because w_{t-1} is almost surely in the column space of $V_{s_{t-1}}$. □

Proof of Theorem 2. Slight differences in assumptions concerning shocks and timing conventions keep us from directly appealing to the results in Costa et al. [6], but the proof below is based on their treatment. See Chapter 3, Theorems 3.9 and 3.33 in particular, for a more expansive treatment of these topics.

Clearly any process y_t defined by Eqs. (15)–(17) is a solution of Eq. (1). So, we must show that any process y_t defined by Eqs. (15)–(17) is MSS if and only if Eq. (19) holds. Since the exogenous process u_t is mean-zero and independent of the Markov process s_t, for any y_t given by Eqs. (15)–(17) we have $E[y_t] = E[w_t]$ and

$$E\left[y_t y'_t\right] = E\left[G_{s_t} u_t u'_t G'_{s_t}\right] + E\left[G_{s_t} u_t \gamma'_t V'_{s_t} V_{s_t}\right] + E\left[V_{s_t} V'_{s_t} \gamma_t u'_t G'_{s_t}\right] + E\left[w_t w'_t\right].$$

Since u_t and γ_t are assumed to be jointly MSS and independent of the ergodic Markov process s_t, the first three terms on the right-hand side of the above equation will converge as t increases without bound. Thus y_t will be MSS if and only if w_t is MSS.

We now show that w_t is MSS if and only if $r_\sigma(M_1(\Phi_{i,j})) < 1$ by first showing that $\lim_{t\to\infty} E[w_t w'_t]$ exists if and only if $M_1(\Phi_{i,j}) < 1$ and then showing that if $M_1(\Phi_{i,j}) < 1$, then $\lim_{t\to\infty} E[w_t]$ exists. Note that

$$E\left[w_t w'_t 1_{\{s_t=j\}}\right] = \sum_{i=1}^{h} p_{i,j} \Lambda_{i,j} E\left[w_{t-1} w'_{t-1} 1_{\{s_{t-1}=i\}}\right] \Lambda'_{i,j} + V_j V'_j E\left[\gamma_t \gamma'_t 1_{\{s_t=j\}}\right] V_j V'_j.$$

Here, we have used the fact that $E[(\gamma_t 1_{\{s_t=j\}})(w'_{t-1} 1_{\{s_{t-1}=i\}})] = 0$, which follows from our assumption that γ_t is mean zero and independent of the Markov process s_t.

Define linear operators $\mathcal{T}$ and $\mathcal{S}$ by

$$\begin{aligned} \mathcal{T}(X_1, \ldots, X_h) &= \left(\sum_{i=1}^{h} p_{i,1} \Lambda_{i,1} X_i \Lambda'_{i,1}, \ldots, \sum_{i=1}^{h} p_{i,h} \Lambda_{i,h} X_i \Lambda'_{i,h}\right), \\ \mathcal{S}(X_1, \ldots, X_h) &= X_1 + \cdots + X_h, \end{aligned}$$

where X_j is an $n \times n$ matrix. Let

$$\Sigma_{j,t} = E\big[w_t w_t' 1_{\{s_t=j\}}\big],$$
$$\Sigma_t = (\Sigma_{1,t}, \dots, \Sigma_{h,t}),$$
$$\Omega_{j,t} = V_j V_j' E\big[\gamma_t \gamma_t' 1_{\{s_t=j\}}\big] V_j V_j',$$
$$\Omega_t = (\Omega_{1,t}, \dots, \Omega_{h,t}).$$

Since γ_t is MSS, $\lim_{t\to\infty} E[\gamma_t \gamma_t'] = \Omega_\gamma$ for some symmetric and positive semi-definite matrix Ω_γ. Since γ_t is independent of the ergodic Markov process s_t,

$$\Omega = \lim_{t\to\infty} \Omega_t = \big(p_1 V_1 V_1' \Omega_\gamma V_1 V_1', \dots, p_h V_h V_h' \Omega_\gamma V_h V_h'\big),$$

where p_j is the ergodic probability that $s_t = j$. The linear operators $\mathcal{T}$ and $\mathcal{S}$ are of interest since $E[w_t w_t'] = \mathcal{S}(\Sigma_t)$, and $\Sigma_t = \mathcal{T}(\Sigma_{t-1}) + \Omega_t$. Iterating, we see that

$$\Sigma_t = \mathcal{T}^t(\Sigma_0) + \sum_{k=1}^{t} \mathcal{T}^{t-k}(\Omega_k).$$

Since $\mathcal{T}$ is a linear operator, it has a matrix representation which is given by

$$\operatorname{vec}\big(\mathcal{T}(X_1, \dots, X_h)\big) = M_1(\Lambda_{i,j}) \operatorname{vec}(X_1, \dots, X_h),$$

where $\operatorname{vec}(\cdot)$ stacks the columns of the matrices into a column vector. So, if $r_\sigma(M_1(\Lambda_{i,j})) < 1$, then

$$\lim_{t\to\infty} E\big[w_t w_t'\big] = \lim_{t\to\infty} \sum_{k=1}^{t} \mathcal{T}^{t-k}(\Omega_k) = \sum_{k=0}^{\infty} \mathcal{T}^k(\Omega),$$

where the last term is a convergent series. On the other hand, if w_t is MSS so that $\lim_{t\to\infty} E[w_t w_t']$ exists and does not depend on the initial condition Σ_0, then it must the case that $\lim_{t\to\infty} \mathcal{T}^t(\Sigma_0) = 0$ for any $\Sigma_0 = (\Sigma_{1,0}, \dots, \Sigma_{h,0})$ such that $\Sigma_{j,0}$ is symmetric and positive semi-definite. However, it was shown in Costa and Fragoso [5] that if $\lim_{t\to\infty} \mathcal{T}^t(\Sigma_0) = 0$ for any $\Sigma_0 = (\Sigma_{1,0}, \dots, \Sigma_{h,0})$ such that $\Sigma_{j,0}$ is symmetric and positive semi-definite, then it is the case that $\lim_{t\to\infty} \mathcal{T}^t(\Sigma_0) = 0$ for *any* Σ_0, which implies that $r_\sigma(M_1(\Lambda_{i,j})) < 1$. Because

$$M_1(\Lambda_{i,j}) = \operatorname{diag}(V_i \otimes V_i) M_1(\Phi_{i,j}) \operatorname{diag}\big(V_i' \otimes V_i'\big),$$

the non-zero eigenvalues of $M_1(\Lambda_{i,j})$ are the same as the non-zero eigenvalues of $M_1(\Phi_{i,j})$, which implies $r_\sigma(M_1(\Lambda_{i,j})) = r_\sigma(M_1(\Phi_{i,j}))$. Thus we have shown that $\lim_{t\to\infty} E[w_t w_t']$ exists if and only if $M_1(\Phi_{i,j}) < 1$.

All that remains to be shown is that if $M_1(\Phi_{i,j}) < 1$, then $\lim_{t\to\infty} E[w_t]$ exists. Consider the homogeneous system $\hat{w}_t = \Lambda_{s_{t-1},s_t} \hat{w}_{t-1}$. It is easy to see that $E[w_t] = E[\hat{w}_t]$ and $E[\hat{w}_t \hat{w}_t'] = \mathcal{T}^t(\Sigma_0)$. Since

$$0 \leqslant E[w_t]E[w_t]' = E[\hat{w}_t]E[\hat{w}_t]' \leqslant E\big[\hat{w}_t \hat{w}_t'\big] = \mathcal{T}^t(\Sigma_0),$$

if $M_1(\Phi_{i,j}) < 1$, then $\lim_{t\to\infty} E[w_t] = 0$.[7] □

Proof of Proposition 1. If we define $V_i = v_i/\|v_i\|$ and $\Phi_{i,j} = \|v_j\| c_i/\|v_i\|$, then Eq. (17) can be written in matrix form as

[7] For square matrices X_1 and X_2, $X_1 \leqslant X_2$ if and only if $X_2 - X_1$ is positive semi-definite.

$$\left(\begin{bmatrix} \Gamma_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \Gamma_h \end{bmatrix} - \begin{bmatrix} p_{1,1}\Phi_{1,1} & \cdots & p_{1,h}\Phi_{1,h} \\ \vdots & \ddots & \vdots \\ p_{h,1}\Phi_{h,1} & \cdots & p_{h,1}\Phi_{h,1} \end{bmatrix}\right) \begin{bmatrix} v_1/\|v_1\| \\ \vdots \\ v_h/\|v_h\| \end{bmatrix} = 0,$$

which is equivalent to Eq. (20). Thus, by Theorem 2, the solution given by Eqs. (22) and (23) will be MSS if and only if $r_\sigma(M_1(\Phi_{i,j})) < 1$. Since

$$M_1(\Phi_{i,j}) = \operatorname{diag}\big(\|v_i\|^2\big)\big(\operatorname{diag}\big(c_i^2\big)P\big)\operatorname{diag}\big(\|v_i\|^2\big)^{-1},$$

$r_\sigma(M_1(\Phi_{i,j})) = r_\sigma(\operatorname{diag}(c_i^2)P)$. This completes the proof of the first two parts of the proposition.

To prove the third part, note that the process y_t given by Eqs. (15) and (16) will be bounded if and only the process w_t given by Eq. (16) is bounded. Since $\Lambda_{s_{t-1},s_t} = c_{s_{t-1}} v_{s_t} v'_{s_{t-1}}/\|v_{s_{t-1}}\|^2$,

$$\begin{aligned} \|w_t\| &= \|v_{s_t}\| \left| \prod_{i=1}^{t} c_{s_{i-1}} \frac{v'_{s_0} w_0}{\|v_{s_0}\|^2} + \sum_{i=2}^{t} \left(\prod_{j=i}^{t} c_{s_{j-1}} \right) v'_{s_{i-1}} \gamma_{i-1} + v'_{s_t} \gamma_t \right| \\ &\leqslant \|v_{s_t}\| \left(\sum_{i=0}^{t} c^i a \right) \end{aligned}$$

where $c = \max\{|c_1|, \ldots, |c_h|\}$ and $a = \sup\{|v'_{s_t}\gamma_t|, |v'_{s_0} w_0|/\|v_{s_0}\|^2\}$. Thus the w_t, and hence the y_t, will be bounded if $|c_i| < 1$ for all i.

On the other hand, if $|c_i| \geqslant 1$ for some i, then for as long as the Markov process remains in state i, w_t will grow exponentially ($|c_i| > 1$) or follow a random walk ($|c_i| = 1$). Since the Markov process can remain in state i for arbitrarily long periods of time, the process w_t, and hence the process y_t, cannot be bounded. □

References

[1] Andrew P. Blake, Fabrizio Zampolli, Optimal monetary policy in Markov-switching models with rational expectations agents, Bank of England, Working Paper No. 298, June 2006.

[2] Richard Clarida, Jordi Galí, Mark Gertler, Monetary policy rules and macroeconomic stability: Evidence and some theory, Quart. J. Econ. CXV (2000) 147–180.

[3] Timothy Cogley, Thomas J. Sargent, Evolving US post-World War II inflation dynamics, NBER Macroeconomics Annual 16 (2002) 331–373.

[4] Timothy Cogley, Thomas J. Sargent, Drifts and volatilities: Monetary policies and outcomes in the post WWII U.S., Rev. Econ. Dynam. 8 (April 2005) 262–302.

[5] O.L.V. Costa, M.D. Fragoso, Comments on 'stochastic stability of jump linear systems', IEEE Trans. Automat. Control 49 (8) (August 2004) 1414–1416.

[6] O.L.V. Costa, M.D. Fragoso, R.P. Marques, Discrete-Time Markov Jump Linear Systems, Springer, New York, 2004.

[7] Troy Davig, Eric M. Leeper, Fluctuating macro policies and the fiscal theory, in: Daron Acemoglu, Kenneth Rogoff, Michael Woodford (Eds.), NBER Macroeconomic Annual 2006, MIT Press, Cambridge, MA, 2006.

[8] Troy Davig, Eric M. Leeper, Generalizing the Taylor principle, Amer. Econ. Rev. 97 (3) (June 2007) 607–635.

[9] Roger E.A. Farmer, Daniel F. Waggoner, Tao Zha, Minimal state variable solutions to Markov-switching rational expectations models, Federal Reserve Bank of Atlanta, Working Paper 2008-23, October 2008.

[10] Roger E.A. Farmer, Daniel F. Waggoner, Tao Zha, Generalizing the Taylor principle: A comment, Amer. Econ. Rev. (2009), forthcoming.

[11] Roger E.A. Farmer, Daniel F. Waggoner, Tao Zha, Indeterminacy in a forward looking regime switching model, Int. J. Econ. Theory 5 (2009) 69–84.

[12] James D. Hamilton, A new approach to the economic analysis of nonstationary time series and the business cycle, Econometrica 57 (2) (1989) 357–384.

[13] Eric M. Leeper, Equilibria under 'active' and 'passive' monetary and fiscal policies, J. Monet. Econ. 27 (1991) 129–147.

[14] Eric M. Leeper, Tao Zha, Modest policy interventions, J. Monet. Econ. 50 (8) (2003) 1673–1700.

[15] Zheng Liu, Daniel F. Waggoner, Tao Zha, Asymmetric expectation effects of regime shifts in monetary policy, Rev. Econ. Dynam. 12 (2) (April 2008) 284–303.

[16] Thomas A. Lubik, Frank Schorfheide, Computing sunspot equilibria in linear rational expectations models, J. Econ. Dynam. Control 28 (2003) 273–285.

[17] Thomas A. Lubik, Frank Schorfheide, Testing for indeterminacy: An application to U.S. monetary policy, Amer. Econ. Rev. 94 (1) (2004) 190–219.

[18] Bennett T. McCallum, On non-uniqueness in rational expectations models: An attempt at perspective, J. Monet. Econ. 11 (March 1983) 139–168.

[19] Christopher A. Sims, Tao Zha, Were there regime switches in US monetary policy? Amer. Econ. Rev. 96 (2006) 54–81.

[20] Lars E.O. Svensson, Noah Williams, Monetary policy with model uncertainty: Distribution forecast targeting, manuscript, Princeton University, September 2005.