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A two-country model of endogenous growth

Roger E.A. Farmer [a,b,*], Amartya Lahiri [c]

[a] *Department of Economics, 8283 Bunche Hall, UCLA, Los Angeles, CA 90095-1477, USA*
[b] *CEPR, UK*
[c] *Federal Reserve Bank of New York, New York, USA*

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Abstract

We study the competitive equilibria of a two-country endogenous growth model in which the source of growth is the linearity of technology in reproducible inputs. We begin by showing that in a model with no externalities there is a unique equilibrium; however, there are multiple ways in which the social planner can allocate production plans across countries. We then introduce an externality to human capital and we show that the model has multiple equilibria that can be Pareto ranked. In many of these equilibria there are perfectly foreseen discrete reallocations of capital from one country to another, accompanied by discrete jumps in growth rates.

1. Introduction

In this paper we study a two-country version of a standard two-sector model of endogenous growth analyzed by Lucas (1988) with and without human capital externalities. We show that in the model without externalities the equilibrium is unique but the distribution of production across the two countries is indeterminate. In the presence of externalities, the model gives rise to multiple equilibria that can be Pareto ranked. In particular, the model with externalities has three types of equilibria. First, there is a symmetric equilibrium in which both countries invest and grow at the same rate. Second, there are two asymmetric

* Corresponding author.
E-mail addresses: rfarmer@econ.ucla.edu (R.E.A. Farmer), amartya.lahiri@ny.frb.org (A. Lahiri).

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equilibria in which one of the two countries shuts down its human capital sector completely and, hence, doesn't grow at all. Third, there is a *continuum* of switching equilibria in which an asymmetric equilibrium prevails for a finite length of time followed by a switch to the symmetric equilibrium. Crucially, the existence of multiple equilibria in the model *does not* depend on the size of the production externality: an arbitrarily small externality is sufficient to guarantee the existence of multiple equilibria.

In the absence of externalities, our model is an open economy version of the two-sector convex model of endogenous growth which, to our knowledge, was first studied by Uzawa (1965). Lucas added externalities to the Uzawa framework in his 1988 paper (Lucas, 1988) on economic development, and a more detailed version of the model that allows for general production technologies has been studied by Bond et al. (1996). Previous related convex models of economic growth include Jones and Manuelli (1990), Rebelo (1991) and our own previous working paper (Farmer and Lahiri, 2004) in which we examined the implications of an open-economy version of the Bond–Wang–Yip model for trade and growth (Bond et al., 1996). The current paper builds on our previous analysis by studying the equilibria that arise in this model in the presence of externalities.

It has been known for some time that two-sector models of growth may lead to indeterminacies in the presence of externalities. Boldrin and Rustichini (1994) analyze a series of one-capital-good models both with bounded and unbounded growth, and Chamley (1993) and Benhabib and Perli (1994) demonstrate the existence of indeterminate equilibria in versions of the two-sector Uzawa–Lucas model. Our work is closely related to these papers but the mechanism by which indeterminacy occurs is very different: It relies critically on the assumption that there are two alternative geographical locations in which a common traded good may be produced. Unlike the examples in Chamley (1993) and Benhabib and Perli (1994), in our two-country two-sector model multiple equilibria may arise in the presence of arbitrarily small externalities.

2. A simple example

The main idea of our paper can be illustrated in a simple AK model of endogenous growth. Suppose that there are two countries and two technologies

$$Y = AK,$$
$$Y' = A'K',$$
$$K + K' = W,$$

where Y and Y' represent output in countries 1 and 2, W is the stock of world wealth and K and K' represent the stocks of wealth used for production in each country. Notice first that if both countries use the same technology, that is if $A = A'$, then the location of production is indeterminate. In this model the welfare theorems hold and one can represent competitive equilibria as the solutions to a social planning problem. Notice that the social planner does not care in which country production is located since the path of output would be identical whichever technology was used.

Now suppose that there are externalities in production represented by the conditions

$$A = \overline{K}^{\gamma},$$
$$A' = \overline{K}'^{\gamma},$$

where $\overline{K}$ and $\overline{K}'$ represent aggregate capital in each country. In this case the competitive equilibria of the economy no longer solve a planning problem. This model has three equilibria. In a symmetric equilibrium capital is equally located across countries, and

$$A = A' = \frac{W}{2}.$$

The other two equilibria are asymmetric, in which either

$$A = W > A' = 0$$

or

$$A' = W > A = 0.$$

These asymmetric equilibria are more efficient than the symmetric equilibrium since they exploit the non-convexity in production arising from production externalities. The main idea of this paper is that this example can be generalized to the Lucas (1988) model of endogenous growth in a two-country world and that this generalization has important implications for world income distribution.

3. The model

We begin by studying a social planning problem. We assume that the social planner maximizes the discounted sum of utilities of two representative agents weighted with the welfare weight b,

$$\max V = \int_{t=0}^{\infty} \mathrm{e}^{-\rho t}\big[bU(C) + (1-b)U'(C')\big]\,\mathrm{d}t. \tag{1}$$

In this equation C and C' represent consumption of the agent in country 1 and 2 respectively and ρ is the common rate of time preference. We further specialize the utility function to take the form

$$U(C) = \log(C), \qquad U'(C') = \log(C').$$

We conjecture that our results will generalize to preferences that are homogeneous in consumption but not beyond this class since homogeneity is required for balanced growth.

We allow for perfect mobility of capital but zero mobility of labor and we assume that the social planner faces the following constraints

$$\dot{W} = F(K, HQu) + F(W-K, H'Q'u') - C - C', \tag{2}$$
$$\dot{H} = \delta H(1-u), \tag{3}$$
$$\dot{H}' = \delta H'(1-u'), \tag{4}$$
$$W(0) = W_0, \quad H(0) = H_0, \quad H'(0) = H_0'. \tag{5}$$

The notation $F(K, HQu)$ refers to the final goods technology which we assume to be strictly concave and common across countries. We let u and u' refer to the fractions of labor allocated to final goods production in each country. Equations (3) and (4) represent the technologies in each country for producing human capital; the variables $(1-u)$ and $(1-u')$ represent the fractions of labor allocated to production of human capital and the parameter δ is a constant that represents the productivity of the human capital technology. Throughout the paper we maintain the assumption $\rho < \delta$. This is a necessary condition if the model is to exhibit endogenous growth. If it fails, the equilibria of the model will be characterized by stagnation and the stocks of physical and human capital will converge asymptotically to zero.

The variables Q and Q' refer to human-capital augmenting technical progress. To capture the idea of knowledge spillovers in the process of human capital acquisition, we assume that they are determined by the equations

$$Q = \overline{H}^{\gamma}, \qquad Q' = \overline{H}'^{\gamma},$$

where $\overline{H}$ and $\overline{H}'$ represent the economy wide levels of human capital in each country and γ is a parameter that indexes the importance of externalities.

3.1. A pseudo planning problem

Following Kehoe et al. (1992) we model equilibria in an economy with externalities by studying a *pseudo planning problem*. By this we mean that the social planner solves Problem 1 taking constraints (2)–(5) as given and, importantly, we assume that the social planner ignores the existence of externalities by taking the variables Q and Q' to be exogenous. Given a solution for given Q and Q' we solve a fixed point problem by letting $Q = H^{\gamma}$ and $Q' = H'^{\gamma}$. Kehoe et al. show that every solution to a fixed point problem of this kind can be decentralized as a competitive equilibrium by a suitable choice of the initial wealth distribution. Since the solutions to a pseudo-social planning problem can be decentralized as competitive equilibria, we refer to them as equilibria. Our purpose is not to study welfare issues but to gain insight into a competitive equilibrium with externalities and by setting up the problem in this way we are able to economize on notation and provide simpler proofs of our main results.

We write the planner's Hamiltonian, where Λ, M and M' are the planner's co-state variables, as follows.

Problem 1.

$$\begin{aligned} \max H = {} & b\log(C) + (1-b)\log C' \\ & + \Lambda\big(F(K, HQu) + F(W-K, H'Q'u') - C - C'\big) \\ & + \delta M H(1-u) + \delta M' H'(1-u'). \end{aligned}$$

Define the following transformed variables:

$$\psi = \frac{M}{\Lambda Q}, \qquad \psi' = \frac{M'}{\Lambda Q'}, \qquad Z = W + HQ\psi + H'Q'\psi', \tag{6}$$

$$c = \frac{C}{Z}, \qquad \lambda = \Lambda Z, \qquad v = \frac{HQ\psi}{Z}, \qquad v' = \frac{HQ\psi'}{Z}, \tag{7}$$

$$k = \frac{K}{HQu}, \qquad k' = \frac{W-K}{H'Q'u'}, \qquad f(k) = F\left(\frac{K}{HQu}, 1\right). \tag{8}$$

The variable ψ represents the shadow price of human capital in country 1 and Z represents the value of world wealth. Throughout the paper, if x is the value of some variable in country 1, we use the notation x' to mean the value of the same variable in country 2.

Notice that in defining world wealth we have multiplied human wealth by Q; this reflects the assumption that human wealth is measured in efficiency units. Individual agents recognize that Q grows but they do not attribute the source of this growth to their own actions. The variable v represents the ratio of the value of human wealth to world wealth and k is the ratio of physical to human wealth located in country 1. λ is the shadow price of consumption.

We also define the following functions:

$$f_H(k) = f(k) - kf_k(k), \qquad \phi(\hat{\psi}) = f_H^{-1}(\delta\hat{\psi}), \quad \hat{\psi} = \max[\psi, \psi']. \tag{9}$$

The function $f_H(k)$ is the marginal product of human capital as a function of the capital/labor ratio. The function $\phi(\hat{\psi})$ is the capital/labor ratio at which a country would find it profitable to produce both human and physical capital if the relative price of capital were equal to $\hat{\psi}$. If both countries produce human capital then $\hat{\psi} = \psi = \psi'$. In words this means that the shadow price of human capital will be the same in both countries. But it is also possible that ψ may not equal ψ' and, in that case, human capital will not be produced in the country with the lower value of human capital. We discuss these possibilities further below. Table 1 defines our transformed variables for future reference.

Our first proposition characterizes the properties of an equilibrium by laying out the necessary and sufficient conditions for maximization of the pseudo-social planning problem. The only novel aspect of the proposition is the use of the variable Z to represent

Table 1
Definitions of variables

Z	World wealth (physical plus the value of human capital).
W	World stock of physical capital.
C	Consumption in country 1 (a prime on a variable denotes country 2).
K	Stock of physical capital in country 1.
H	Stock of human capital in country 1.
HQ	Human capital in efficiency units in country 1.
u	Fraction of labor in country 1 allocated to physical capital production.
$c = \frac{C}{Z}$	The ratio of consumption in country 1 to world wealth.
$\omega = \frac{W}{Z}$	The ratio of world physical wealth to total world wealth.
Q	Productivity (taken as exogenous by individual agents).
ψ	The relative price of human capital.
$\hat{\psi}$	The higher of the relative price of H in the two countries.
$k = \frac{K}{HQu}$	The capital/labor ratio in the physical goods sector in country 1.
$v = \frac{\psi QH}{Z}$	The value of human capital in efficiency units as a fraction of world wealth.
λ	The shadow value of c (also of c').

world wealth. By choosing variables that are ratios to Z, we are able to develop a simple characterization of balanced growth paths in a world in which agents have logarithmic preferences. Typically, the system of differential equations that describes convergence to a balanced growth path would be third order. By choosing the ratio of consumption to wealth to be one variable of the system we are able to reduce the order of the system by 1. This reduction occurs since the stable manifold of a stationary equilibrium has the property that the ratio of C to Z is a constant.

The necessary and sufficient conditions for dynamic optimization imply that any solution to Problem 1 must solve the following set of static first-order conditions:

$$\frac{b}{c} = \frac{(1-b)}{c'} = \lambda, \tag{10}$$

$$f_k(k) = f_k(k'), \tag{11}$$

$$(1-u)\big(f_H(k) - \delta\psi\big) = 0, \quad (1-u) \geqslant 0, \quad \big(f_H(k) - \delta\psi\big) \geqslant 0, \tag{12}$$

$$(1-u')\big(f_H(k') - \delta\psi'\big) = 0, \quad (1-u') \geqslant 0, \quad \big(f_H(k') - \delta\psi'\big) \geqslant 0, \tag{13}$$

$$1 = v + v' + \phi\big(\hat{\psi}\big)\left(\frac{uv}{\psi} + \frac{u'v'}{\psi'}\right), \tag{14}$$

together with the dynamic equations,

$$\frac{\dot{Z}}{Z} = f_k(k) - \frac{1}{\lambda}, \tag{15}$$

$$\frac{\dot{\lambda}}{\lambda} = \rho - \frac{1}{\lambda}, \tag{16}$$

$$\frac{\dot{\psi}}{\psi} = f_k(k) - \frac{u f_H(k)}{\psi} - (1-u)\delta(1+\gamma), \tag{17}$$

$$\frac{\dot{\psi}'}{\psi'} = f_k(k') - \frac{u' f_H(k')}{\psi'} - (1-u')\delta(1+\gamma), \tag{18}$$

$$\frac{\dot{v}}{v} = \frac{-u f_H(k)}{\psi} + \frac{1}{\lambda}, \tag{19}$$

$$\frac{\dot{v}'}{v'} = \frac{-u' f_H(k')}{\psi'} + \frac{1}{\lambda}, \tag{20}$$

and the transversality conditions

$$\lim_{T\to\infty} \mathrm{e}^{-\rho T} \Lambda W = 0, \qquad \lim_{T\to\infty} \mathrm{e}^{-\rho T} MH = 0, \qquad \lim_{T\to\infty} \mathrm{e}^{-\rho T} M'H' = 0. \tag{21}$$

Equation (10) is the first-order condition for the choice of consumption and (11) ensures that capital is efficiently allocated across countries. Conditions (12) and (13) are the Kuhn–Tucker conditions for allocation of labor across industries in the two countries. If u' is equal to 1 then the production of human capital shuts down in country 2. We explicitly allow for the possibility that this may occur in equilibrium. Equation (15) gives the evolution of world wealth.

In characterizing the properties of an equilibrium, it is helpful to make use of the variable $\hat{\psi}$ defined as

$$\hat{\psi} = \max\{\psi, \psi'\},$$

where ψ and ψ' are the relative prices of human to physical capital in the two countries. In an interior equilibrium in which human capital is produced in both countries these two prices would be equated. But there may also exist equilibria in which one or other of these prices is higher than the other. These possibilities are reflected in Eq. (14), which is derived by dividing Eq. (22)

$$K + K' = W \tag{22}$$

by Z, and combining the resulting expression with the definitions of v, v', and ϕ, to arrive at

$$\phi(\psi)\frac{vu}{\psi} + \phi(\psi')\frac{vu'}{\psi'} = 1 - v - v'. \tag{23}$$

If ψ and ψ' are unequal then human capital production in country 1 or 2 will cease but the price of existing human capital in that country will still be well defined. Even if human capital production shuts down, existing human capital will still be used to produce physical output in that country and the capital/labor ratio will be equated across physical goods industries. It is this insight that allows us to replace $\phi(\psi)$ and $\phi(\psi')$ by $\phi(\hat{\psi})$ in Eq. (23) to derive Eq. (14) which is reproduced below:

$$\phi(\hat{\psi})\left(\frac{vu}{\psi} + \frac{vu'}{\psi'}\right) + v + v' = 1.$$

Equation (14) is a constraint that links the value of human capital in the two countries to relative prices.

To solve growth models, it is often useful to find a change of variable such that the transformed variables are stationary along a balanced growth path. We chose to normalize consumption and capital by dividing each variable by Z, a variable that represents "world wealth." The choice of Z as a normalization variable is especially convenient in a model in which agents have logarithmic preferences since these preferences imply that the consumption wealth ratio, c, is constant at all times. This allows us to reduce the order of the dynamical system that describes equilibrium trajectories.

Definition 1. An interior solution to Problem 1 is a solution for which

$$0 < u < 1, \qquad 0 < u' < 1.$$

An interior solution is one in which both industries operate in both countries at all points in time. Since Problem 1 is concave in C and C', there will be a unique solution for the consumption allocation. But the Lagrangian is only weakly concave in K. Later on, we will show that this fact implies that there may be many production plans, indexed by different time paths for u, that implement the unique consumption allocation.

We next demonstrate that a conditional version of factor price equalization holds in our model.

Proposition 1 (Conditional factor price equalization). *In an interior solution to Problem* 1, $\psi = \psi'$; *that is, relative factor prices adjusted for productivity differences will be equated across countries at all times.*

Proof. From the Kuhn–Tucker conditions, (12) and (13), it follows that for interior u and u' we must have

$$f_H(k) = \delta\psi, \qquad f_H(k') = \delta\psi'. \tag{24}$$

From Eq. (11) we know that $k = k'$. Hence, it must also be true that $\psi = \psi'$. $\square$

Since ψ is defined as $M/(\Lambda Q)$ and ψ' is defined as $M'/(\Lambda' Q')$, ψ and ψ' measure the relative prices of physical to human capital in the two countries after adjusting for labor productivity, Q and Q'. Hence, $\psi = \psi'$ implies the equalization of factor prices adjusted for labor productivity. We refer to this as *conditional factor price equalization*. However, in general $Q\psi \neq Q'\psi'$ unless $Q = Q'$. Hence, unconditional factor price equalization fails to arise in general along an interior solution.[1]

4. Equilibria in the absence of externalities

In this section we study equilibria for the case of no externalities, $\gamma = 0$ and hence productivity (equal to Q) is constant and equal to 1. We show that equilibria can be characterized as the solution to a system of differential equations in three state variables. Two of these variables, ω and ψ describe convergence to the balanced growth path and they form an autonomous subsystem. The third variable, Z describes the evolution of the growth rate.

It is straightforward to check that for the case $Q = 1$, if there exists an interior solution to Problem 1, the variables ω, ψ and ψ' form an autonomous subsystem with dynamics governed by the following equations:

$$\frac{\dot{\psi}}{\psi} = f_k\big(\phi(\psi)\big) - \delta, \tag{25}$$

$$\psi = \psi', \tag{26}$$

$$\frac{\dot{\omega}}{\omega} = \frac{\delta\psi}{\phi(\psi)} - \frac{\rho}{\omega} + \rho. \tag{27}$$

It can be verified that the subsystem (25),(27) has a unique stationary solution $\bar{\psi}, \bar{\omega}$ that satisfies the equations

$$\bar{\psi} = \frac{f_H[f_k^{-1}(\delta)]}{\delta}, \qquad \bar{\omega} = \frac{\rho\phi(\bar{\psi})}{\delta\bar{\psi} + \rho\phi(\bar{\psi})}. \tag{28}$$

[1] This is consistent with the empirical findings of Treffler (1993), who shows that unconditional factor price equalization fails to hold but that conditional factor price equalization is a relatively good characterization of the data.

The variables k, k', c, c' and Z can be derived from the steady state versions of Eqs. (15)–(20).

The dynamical system (25) and (27) when linearized around the steady state is characterized by two roots of opposing signs. Hence, for initial values of H, H' and W such that[2]

$$\frac{H_0 + H_0'}{W_0} \in N\left(\frac{1-\bar{\omega}}{\bar{\omega}\bar{\psi}}\right), \tag{29}$$

there exists a one-dimensional manifold of solutions to the dynamical system (25) and (27) that converges asymptotically to $\{\bar{\omega}, \bar{\psi}\}$.

The following proposition establishes that there is not just one but many ways of allocating production across countries in this economy. Multiplicity arises as a consequence of the weak concavity of the pseudo planning problem in the allocation of capital across countries.

Proposition 2. *For initial values of H_0, H_0', W_0 that satisfy conditions* (29)*, there exists a continuum of interior solutions to Problem* 1.

Proof. Consider the path for v, u, ω and ψ given by the equations

$$u = \frac{\rho}{\delta}, \qquad v = v_0, \qquad v' = 1 - v - \omega,$$
$$u' = \left(\frac{\omega\psi}{\phi(\psi)} - uv\right)\frac{1}{v'},$$

where ω and ψ are determined as solutions to (25) and (27) and ψ_0 is chosen such the system begins on the stable manifold. Since this solution satisfies the transversality conditions, there exists at least one interior solution to Problem 1. To establish that there is a continuum of alternative solutions, let u equal some arbitrary value $\tilde{u}$ for an interval of time $t \in [0, T)$ and let $u = \rho/\delta$, thereafter, $t \in [T, \infty]$. Since this choice of u guarantees that v, v', ω, ψ and ψ' converge to positive finite numbers, transversality is satisfied and hence, by arbitrary choice of $\tilde{u}$, arbitrarily close to any one solution there is another. □

In the plan in which $u = \rho/\delta$ for all t, the Social Planner chooses to maintain the relative stocks of human capital to world wealth, equal to their initial levels. But this is not the only way to implement a given consumption plan. As an alternative, the Planner could choose to build up human capital in country 1 by lowering u and allocating more labor time in that country to the production of human capital. To maintain the same production of physical capital the Planner would reallocate labor in country 2 out of human capital production and into physical capital production. During this process the stock of human capital in country 1 will be increasing relative to the stock in country 2. If at date T resources are switched back to human capital production in country 1, the ratio of human capital to world wealth (denoted v_T) will be constant from that date on.

[2] We use the notation $N(x)$ to mean an epsilon neighborhood of x.

The indeterminacy of the location of production also implies indeterminacy of the pattern of capital flows across countries. If two autarkic economies with different relative factor prices, i.e., $\psi \neq \psi'$, were to open up to trade, then the relative factor price ratios would immediately be equated. But this factor price equalization would not imply an equalization of the physical to human capital ratios across countries since the pattern of capital flows is indeterminate. Any change in the relative factor price ratio (both at the time of opening up to trade as well as over time) can be accommodated through a sectoral reallocation of factors rather than through a cross-country movement of capital.

5. Equilibria with externalities

In the economy without externalities, there exists a continuum of production plans; but they are all welfare equivalent since alternative production plans leave the path of world consumption unaffected. In this section we show that this is not the case when there are production externalities in the model. In particular, we set $\gamma > 0$.

To characterize equilibria in the economy with externalities it is convenient to introduce a new variable. We define

$$s = \frac{v'\psi}{v\psi'} \equiv \left(\frac{H'}{H}\right)^{1+\gamma},$$

to be the ratio of human capitals in the two countries, raised to the power $1+\gamma$. We will establish that the world economy can be characterized by a system of differential equations in five variables, ψ, ψ', s, v and Z. The first four of these equations form an autonomous system and the fifth, the $\dot{Z}$ equation, determines the growth rate. We will show that there exist two types of equilibria. In a symmetric equilibrium $\psi = \psi'$ and $\dot{s} = 0$ for all t. In this case the dynamics of an approach to a balanced growth path is characterized by a two-dimensional subsystem in ψ and v. In an asymmetric equilibrium $\psi > \psi'$ for all time and the production of human capital is shut down in country 2. In this case the dynamics of an approach to the balanced growth path is characterized by a four variable dynamics system in s, ψ', v and v'.

5.1. Interior equilibria

In Proposition 1 we established, for the economy with no externalities, that conditional factor price equalization, $\psi = \psi'$, holds at all points in time in an interior equilibrium. We begin by establishing a stronger version of this proposition for the economy with externalities. Specifically we show that conditional factor price equalization holds in the economy with externalities *and* that this implies a unique allocation of factors across industries.

Proposition 3 (Strong factor price equalization). *In any interior solution to Problem* 1 *with positive externalities, relative factor prices and labor allocated to each industry will be equated across countries. That is,* $\psi = \psi'$ *and* $u = u'$.

Proof. From the Kuhn–Tucker conditions, (12) and (13), it follows that for interior u and u' we must have $f_H(k) = \delta\psi$ and $f_H(k') = \delta\psi'$. From Eq. (11) we also know that $k = k'$. Hence, it must also be true that $\psi = \psi'$. Combining (17), (18), (12) and (13) gives

$$\frac{\dot{\psi}}{\psi} = f_k(k) - \delta(1+\gamma) + u\delta\gamma, \qquad \frac{\dot{\psi}'}{\psi'} = f_k(k) - \delta(1+\gamma) + u'\delta\gamma,$$

and since $\psi = \psi'$ for all t, it must be that $u = u'$. □

In the case of no externalities the second part of Proposition 3 does not hold since the differential equations governing the evolution of ψ and ψ' are independent of u and u'. In other words, if there are no externalities then factor price equalization does not imply a unique allocation of resources across industries.

We now turn our attention to establishing the determinants of the relative size of output per capita in the two countries as determined by s, the ratio of human capital in country 1 to country 2 raised to the power $1+\gamma$. We show that this ratio, in an interior equilibrium, is determined by initial conditions. Hence the model implies that in a symmetric interior equilibrium, rich and poor countries grow at the same rate and relative wealth does not change over time.

Proposition 4. *In any interior equilibrium with positive externalities, $s = (H_0'/H_0)^{1+\gamma}$ is a positive constant.*

Proof. We seek to show that $\dot{H} = \dot{H}'$ for all time. From Eqs. (3) and (4),

$$\dot{H} = \delta H(1-u), \qquad \dot{H}' = \delta H(1-u'),$$

and from Proposition 3, $u = u'$; hence s is constant in an interior equilibrium with externalities. □

Our next proposition characterizes the path by which the economy converges to a balanced growth path as the solution to a pair of autonomous differential equations in ψ, the relative price of human capital and v, the value of human capital in country 1 as a fraction of world wealth.

Proposition 5. *For the case of positive externalities, $Q > 1$, an interior equilibrium is characterized by the following pair of autonomous differential equations*:

$$\frac{\dot{\psi}}{\psi} = f_k\big(\phi(\psi)\big) - \delta(1+\gamma) + \delta\gamma u, \tag{30}$$

$$\frac{\dot{v}}{v} = -u\delta + \rho, \tag{31}$$

where u is given by the expression

$$u = \frac{\psi(1 - v(1+s))}{\phi(\psi)v(1+s)}, \tag{32}$$

and

$$s = \left(\frac{H_0'}{H_0}\right)^{1+\gamma} \tag{33}$$

is a positive constant determined by the initial levels of human wealth in the two countries. The variables $k, k', c, c', \psi', v', \omega$ and Z are governed by the equations

$$k = \phi(\psi), \tag{34}$$

$$k' = \phi(\psi), \tag{35}$$

$$c = b\rho, \tag{36}$$

$$c' = (1-b)\rho, \tag{37}$$

$$\psi' = \psi, \tag{38}$$

$$v' = sv, \tag{39}$$

$$\omega \equiv 1 - v - v', \tag{40}$$

$$\frac{\dot{Z}}{Z} = f_k\big(\phi(\psi)\big) - \rho. \tag{41}$$

Proof. Equations (30) and (31) follow directly from (17) and (19) together with the Kuhn–Tucker condition (12) which implies that $\psi = \phi(\psi)$ in a interior equilibrium. Equation (32) follows from (14), Proposition 3, and the definition of s. Equation (33) follows from Proposition 4, Eqs. (34) and (35) are implied by the fact that the equilibrium is interior, Eqs. (36) and (37) follow from (10) and the fact that $\lambda^{-1} = \rho$ is constant (from transversality). Equation (38) is from the fact that equilibrium is interior, Eq. (39) is from Proposition 4 and the definition of s, Eq. (40) is an identity and Eq. (41) follows from (15). □

We have chosen to characterize the system with externalities as a differential equation system in ψ and v rather than in ψ and ω since it will facilitate comparison between symmetric and asymmetric equilibria. These two formulations are closely linked since it is identically true that

$$\omega = 1 - v - v',$$

and in a symmetric equilibrium $v' = sv$ where s is a constant determined by the initial distribution of human capital. For the case of no externalities, ω is a better choice of variable to characterize equilibrium since v and v' are indeterminate. But for the case of an economy with externalities, v is a more convenient choice of variable. In this case, in an interior equilibrium, v and v' are determinate and, as we will see below, the choice of ψ and v as state variables allows comparison with the asymmetric equilibria that are characterized by a four variable differential equation system in s, ψ, ψ' and v.

The following proposition establishes that the Inada conditions are sufficient to imply the existence of a unique balanced growth path.

Proposition 6. *If*

$$\lim_{x\to 0} f_k(x) = \infty, \qquad \lim_{x\to\infty} f_k(x) = 0, \qquad (\textit{Inada conditions})$$

the subsystem (30)–(32) *has a unique stationary solution* $\bar{u}$, $\bar{\psi}$, $\bar{v}$ *that satisfies the equations*

$$\bar{u} = \frac{\rho}{\delta},$$
$$\bar{\psi} = \frac{f_H[f_k^{-1}(\delta + \gamma(\delta - \rho))]}{\delta},$$
$$\bar{v} = \frac{\delta\bar{\psi}}{(1+s)(\rho\phi(\bar{\psi}) + \bar{\psi}\delta)}.$$

Proof. Follows from strict concavity of f plus continuity and the Inada conditions. □

Next we establish, in the case of externalities, that for initial conditions in an open neighborhood of the balanced growth path there is a unique equilibrium that converges to balanced growth.

Proposition 7. *For initial values of* H, H' *and* W *such that*

$$\frac{H_0^{1+\gamma} + H_0'^{1+\gamma}}{W_0} \in N\left(\frac{\bar{v}(1+s)}{(1-\bar{v}(1+s))\bar{\psi}}\right),$$

there exists a one-dimensional manifold of solutions to the dynamical system (30) *and* (31) *that converges asymptotically to* $\{\bar{v}, \bar{\psi}\}$.

Proof. Consider the subsystem defined by Eqs. (30) and (31). The Jacobian of this system is given by

$$J = \begin{vmatrix} f_{kk}\phi' + u_\psi\delta\gamma & \delta\gamma u_v \\ -u_\psi\delta & -u_v\delta \end{vmatrix},$$

where u_v and u_ψ denote the partial derivatives of function

$$u(v, \psi) = \frac{\psi(1 - v(1+s))}{\phi(\psi)v(1+s)}$$

with respect to v and ψ. The determinant of J evaluated at $\{\bar{v}, \bar{\psi}\}$ is given by

$$\det(J) = -u_v\delta f_{kk}\phi'\big|_{\{\bar{\psi},\bar{v}\}} < 0,$$

where the sign follows from the definition of ϕ, the concavity of $f(k)$ and the fact that $u_v < 0$. Since $\det(J)$ is equal to the product of the roots of the localized dynamics, one root must be positive and the other negative and the steady state is saddle path stable. It follows that locally there is a one-dimensional manifold of initial conditions that converges asymptotically to the steady state.

We now establish conditions on H, H' and W for the system to begin close to the steady state. When $\psi = \psi'$ and $v' = sv$, we can write ω as

$$\omega = 1 - v - v' = 1 - v(1+s) = \frac{W}{W + \psi(H^{1+\gamma} + H'^{1+\gamma})}.$$

Rearranging, this expression gives

$$\frac{H^{1+\gamma}+H'^{1+\gamma}}{W}=\left(\frac{v(1+s)}{(1-v(1+s))\psi}\right).$$

Since H and H' are predetermined, it follows that once ψ is determined, so are ψ', v' and v. Since $0<v(1+s)<1$, the condition for the system to begin close to the steady state is given by

$$\frac{H_0^{1+\gamma}+H_0'^{1+\gamma}}{W_0}\in N\left(\frac{\bar{v}(1+s)}{(1-\bar{v}(1+s))\bar{\psi}}\right),$$

which is the condition in the statement of the proposition. □

The state of the economy is completely characterized by three variables, W, H and H'. We have established that this economy exhibits a balanced growth path and that for a set of initial conditions in the neighborhood of this path, there exists a locally unique determinate equilibrium that converges to it. We next turn to the possibility that there may exist other kinds of equilibria.

5.2. Asymmetric equilibria

We turn to the case in which one country shuts down its human capital sector forever. This could happen if either $0<u<1$ and $u'=1$ or $u=1$ and $0<u'<1$ for all t. We will concentrate on the case $u'=1$ and hence country 2 ceases to produce human capital although the fact that the model is symmetric implies that there is also another equilibrium in which it is country 1 that follows this route. The following proposition establishes the properties of the dynamics of a solution in the asymmetric case.

Proposition 8. *For the case $Q>1$, any asymmetric solution is characterized by the following set of four autonomous differential equations*:

$$\dot{\psi}=\psi\big[f_k\big(\phi(\psi)\big)-\delta(1+\gamma)+\delta\gamma u\big], \tag{42}$$

$$\dot{\psi}'=\psi' f_k\big(\phi(\psi)\big)-\delta\psi, \tag{43}$$

$$\dot{v}=v[-u\delta+\rho], \tag{44}$$

$$\dot{s}=s(u-1)\delta(1+\gamma), \tag{45}$$

where u is given by the expression

$$u=u(\psi,\psi',v,s)\equiv\frac{\psi}{v\phi(\psi)}-\frac{\psi}{\phi(\psi)}-s\frac{\psi'}{\phi(\psi)}-s. \tag{46}$$

The variables k, k', c, c', ω and Z are governed by the equations

$$k=\phi(\psi), \tag{47}$$

$$k'=\phi(\psi), \tag{48}$$

$$c=b\rho, \tag{49}$$

$$c' = (1-b)\rho, \tag{50}$$

$$\omega \equiv 1 - v - v', \tag{51}$$

$$\frac{\dot{Z}}{Z} = f_k\big(\phi(\psi)\big) - \rho. \tag{52}$$

Proof. Notice first that in an asymmetric equilibrium $u' = 1$ and $0 < u < 1$. Equation (42) follows from (17) and Eq. (43) from (18). Equation (44) follows from (19) and (12). Equation (45) follows from differentiating the definition of s and substituting the equations of motion for ψ, ψ', v (Eqs. (42)–(44)) and the equation of motion for v' (Eq. (20)), and recognizing that

$$\frac{-u' f_H(k')}{\psi'} = -\frac{\delta\psi}{\psi'}.$$

Equation (46) follows from (14) and the fact that $u' = 1$. Equations (47)–(52) are as in the proof of the symmetric case. □

As in the symmetric case, the Inada conditions imply the existence of a unique asymmetric equilibrium.

Proposition 9. *If*

$$\lim_{x\to 0} f_k(x) = \infty, \qquad \lim_{x\to\infty} f_k(x) = 0, \qquad (\textit{Inada conditions})$$

the subsystem (42)–(46) *has a unique stationary solution* $\{\bar{\psi}, \bar{\psi}', \bar{v}, \bar{s}, \bar{u}\}$ *that satisfies the equations*

$$\bar{u} = \frac{\rho}{\delta}, \tag{53}$$

$$\bar{s} = 0, \tag{54}$$

$$\bar{\psi} = \frac{f_H[f_k^{-1}(\delta + \gamma(\delta - \rho))]}{\delta}, \tag{55}$$

$$\bar{\psi}' = \frac{\delta\bar{\psi}}{\delta + \gamma(\delta - \rho)} < \bar{\psi}, \tag{56}$$

$$\bar{v} = \frac{\delta\bar{\psi}}{\phi(\bar{\psi})\rho + \delta\bar{\psi}}. \tag{57}$$

Proof. Follows from solving the steady state conditions (42)–(45). The only point at which existence and uniqueness is an issue is in showing that Eq. (55) has a solution. This follows from strict concavity of f plus continuity and the Inada conditions. □

Since $\psi \neq \psi'$ in the stationary state, Proposition 9 implies that an asymmetric equilibrium is characterized neither by conditional factor price equalization, nor by unconditional factor price equalization. Note that for $\gamma = 0$, Eq. (56) implies that $\bar{\psi}' = \bar{\psi}$. Hence, the absence of factor price equalization is crucially dependent on the presence of externalities.

Now we turn to the existence of non-stationary equilibria that converge to an asymmetric equilibrium. Since the order of the dynamics is higher in the case of asymmetric equilibria, we must find a pair of initial conditions that is in the neighborhood of an asymmetric growth equilibrium. This amounts to choosing initial conditions such that one country is small relative to the other and such that the large country begins with human capital close to the balanced growth equilibrium for a single country world.

Proposition 10. *For initial values of H, H' and W such that*

$$\frac{H_0^{1+\gamma}}{W_0} \in N\left(\frac{\bar{v}}{(1-\bar{v})\bar{\psi}}\right), \tag{58}$$

$$\frac{H_0'}{H_0} \in N(0), \tag{59}$$

there exists a two-dimensional manifold of solutions to the dynamical system (42)–(45) *that converges asymptotically to* $\{\bar{\psi}, \bar{\psi}', \bar{v}, \bar{s}\}$.

Proof. We begin by establishing that the stable manifold has dimension 2 around the steady state. Define the terms u_s, u_v, u_ψ and $u_{\psi'}$ to be the partial derivatives of the function u evaluated at the point $\{\bar{\psi}, \bar{\psi}', \bar{v}, \bar{s}\}$ and note that since $\bar{s} = 0$,

$$u_{\psi'} = 0.$$

Using this fact now write the Jacobian of the system (42)–(45) evaluated at the point $\{\bar{\psi}, \bar{\psi}', \bar{v}, \bar{s}\}$ as follows:

$$J = \begin{vmatrix} [f_{kk}\phi' + \delta\gamma u_\psi]\psi & 0 & \delta\gamma\psi u_v & \delta\gamma\psi u_s \\ f_{kk}\phi'\psi' - \delta & f_k & 0 & 0 \\ -v\delta u_\psi & 0 & -v\delta u_v & -v\delta u_s \\ 0 & 0 & 0 & (u-1)\delta(1+\gamma) \end{vmatrix},$$

where the fourth row contains only one non-zero entry since $\bar{s} = 0$. The characteristic polynomial of J is given by the equation

$$\big[(\bar{u}-1)\delta(1+\gamma) - \lambda\big]\big[f_k - \lambda\big]\big[\lambda^2 + \lambda\big(\bar{v}\delta u_v - [f_{kk}\phi' + \delta\gamma u_\psi]\bar{\psi}\big) - \bar{v}\delta u_v f_{kk}\phi'\bar{\psi}\big] = 0$$

which has roots

$$\lambda_1 = f_k, \qquad \lambda_2 = (\bar{u}-1)\delta(1+\gamma),$$

plus the roots of the polynomial

$$P(\lambda) = \big[\lambda^2 + \lambda\big(\bar{v}\delta u_v - [f_{kk} + u_\psi]\bar{\psi}\big) - v\delta u_v f_{kk}\phi'\psi\big] = 0.$$

Since $0 < \bar{u} < 1$, λ_1 is positive and λ_2 negative. Let the roots of $P(\lambda)$ be equal to λ_3 and λ_4. Since $\lambda_3\lambda_4 = -\bar{v}\delta u_v f_{kk}\phi'\bar{\psi}$ and $u_v < 0$ (from differentiating u), $f_{kk} < 0$ (by concavity) and $\phi' > 0$ (also from concavity of f), it follows that one root of $P(\lambda)$ is positive and the other is negative. Hence it follows that there is locally a two-dimensional manifold of initial conditions that converges asymptotically to the steady state.

We now establish conditions on H, H' and W for the system to begin close to the steady state. Write ω as

$$\omega = 1 - v - v' = 1 - v\left(1 + \frac{\bar{\psi}'\bar{s}}{\bar{\psi}}\right) = \frac{W}{W + (\bar{\psi} H^{1+\gamma} + \bar{\psi}' H'^{1+\gamma})},$$

and since $\bar{s} = 0$,

$$1 - v = \frac{W/H^{1+\gamma}}{W/H^{1+\gamma} + \bar{\psi}}.$$

Rearranging this expression gives

$$\frac{H_0^{1+\gamma}}{W_0} = \left(\frac{\bar{v}}{(1-\bar{v})\bar{\psi}}\right),$$

which is condition (58). Condition (59) follows directly from the requirement that $s_0 = (H_0'/H_0)^{1+\gamma}$ should begin close to its steady state value of 0. $\square$

To prove the existence of a symmetric equilibrium we used the assumption that the initial ratio of human wealth to non-human wealth was close to the ratio of this variable along the balanced growth path. In the case of an asymmetric equilibrium we need two initial conditions. The first is identical to the condition required for the symmetric case. The second condition, $H_0'/H_0 \in N(0)$, is equivalent to assuming that country 2 is "small" relative to country 1. We use both assumptions about the initial state of the economy because we want to appeal to local results in the theory of differential equations. Although we have not been able to prove global stability theorems, we conjecture that these assumptions are much stronger than required for the existence of equilibrium and that more generally, there will exist equilibria, in reasonably parameterized examples, for much larger sets of initial conditions.

Notice that both sets of initial conditions are mutually consistent. We have established that if the world begins with a ratio of human to physical capital that is close to the balanced growth path and that if one economy is much smaller than the other, then there are two possible equilibria. The small economy may grow at the same rate as the large economy and the world may move to a balanced growth path. Or the small economy may cease to grow and, in this case, all future growth will take place in the large economy. Although the small economy will cease to grow; it will continue to produce the physical commodity, in contrast to the one sector AK model that we discussed in Section 2. It is also worth pointing out that the degree of increasing returns to scale does not need to be large in order for there to be asymmetric equilibria; γ must be greater than 0 but can be arbitrarily close to 0.[3]

In the case of externalities, the symmetric and asymmetric equilibria can be Pareto ranked. The asymmetric equilibrium Pareto dominates the symmetric equilibrium since, by concentrating production in a single location, the Social Planner is able to fully exploit the advantages of increasing returns to scale.

[3] Without externalities the model would still exhibit asymmetric production allocations across countries. But these different production plans would have no implications for consumption and welfare.

5.3. Switching equilibria

We now turn to a third kind of equilibrium. We will show that there is an equilibrium in which the world economy follows the asymmetric equilibrium for a *finite* length of time before switching to the symmetric equilibrium permanently. We refer to this case as a *switching equilibrium*.

The idea behind the proof of existence of a switching equilibrium is as follows. Suppose that the path for u' (time allocated to the production of physical capital in country 2) is given by

$$u'_t = \begin{cases} 1 & \text{for } t \in [0, T], \\ u_t & \text{for } t \in (T, \infty). \end{cases} \tag{60}$$

If such a path for u' is to be an equilibrium then the relative prices of human capital in the two countries, ψ and ψ', must be continuous at T. If this were not the case, since ψ and ψ' represent relative prices, there would be arbitrage opportunities along a perfect foresight path. To satisfy the transversality conditions, it must also be true that the system hits the stable manifold of the symmetric equilibrium at date T and that it remains on this manifold for all $t \geqslant T$. In Proposition 5 we established that, along the stable manifold of a symmetric equilibrium, the following two conditions must hold:

$$v' = sv, \tag{61}$$

$$\psi = \psi'. \tag{62}$$

Since neither ψ nor ψ' can jump at time T, v and v' cannot jump either. Hence, Eqs. (61) and (62) describe two terminal conditions that the dynamic path for the system between $t = 0$ and $t = T$ must satisfy.

Proposition 11. *Let $\{\bar{\psi}, \bar{\psi}', \bar{v}, \bar{s}, \bar{u}\}$ represent the steady state of the asymmetric equilibrium defined in Proposition* 9. *Let the system begin with the initial conditions H_0, H'_0 and W_0 such that*

$$\frac{H_0^{1+\gamma}}{W_0} \in N\left(\frac{\bar{v}}{(1-\bar{v})\bar{\psi}}\right), \tag{63}$$

$$\frac{H'_0}{H_0} \in N(0). \tag{64}$$

Then there exists a continuum of switching equilibria, indexed by T, characterized by the following system of equations. For $t \in [0, T]$ the system is determined by the differential equations:

$$\dot{\psi} = \psi\big[f_k\big(\phi(\psi)\big) - \delta(1+\gamma) + \delta\gamma u\big], \tag{65}$$

$$\dot{\psi}' = \psi' f_k\big(\phi(\psi)\big) - \delta\psi, \tag{66}$$

$$\dot{v} = v[-u\delta + \rho], \tag{67}$$

$$\dot{s} = s(u-1)\delta(1+\gamma), \tag{68}$$

where u is given by the expression

$$u = u(\psi, \psi', v, s) \equiv \frac{\psi}{v\phi(\psi)} - \frac{\psi}{\phi(\psi)} - s\frac{\psi'}{\phi(\psi)} - s. \tag{69}$$

From dates $t \in [T, \infty]$, the system is characterized by the equations:

$$\frac{\dot{\psi}}{\psi} = f_k\big(\phi(\psi)\big) - \delta(1+\gamma) + \delta\gamma u, \tag{70}$$

$$\frac{\dot{v}}{v} = -u\delta + \rho, \tag{71}$$

$$\psi' = \psi, \tag{72}$$

$$s = s_T. \tag{73}$$

The variables k, k', c, c', v', ω and Z are governed by the equations

$$k = \phi(\psi), \tag{74}$$

$$k' = \phi(\psi), \tag{75}$$

$$c = b\rho, \tag{76}$$

$$c' = (1-b)\rho, \tag{77}$$

$$v' = sv, \tag{78}$$

$$\omega \equiv 1 - v - v', \tag{79}$$

$$\frac{\dot{Z}}{Z} = f_k\big(\phi(\psi)\big) - \rho. \tag{80}$$

Proof. We established in Proposition 10 that the stable manifold of the asymmetric steady state has dimension 2. Suppose $T = 0$; then the switching equilibrium is trivially equal to the symmetric equilibrium. Now consider an open interval $T \in (0, \overline{T})$, $\psi_T = \psi'_T$ and choose ψ_T and s_T such that $\{\psi_T, v_T\}$ is on the stable manifold of the symmetric equilibrium. There is a unique orbit of the differential equation system (65)–(68) that passes through the point $\{\psi_T, \psi_T, v_T, s_T\}$. Let $\{\psi_0, \psi'_0\}$ be so chosen that this orbit passes through the point

$$1 - v_0\left(1 + \frac{\psi'_0 s_0}{\psi_0}\right) = \frac{W_0}{W_0 + (\psi_0 H_0^{1+\gamma} + \psi'_0 H_0'^{1+\gamma})},$$

$$\left(\frac{H'_0}{H_0}\right)^{1+\gamma} = s_0.$$

It remains to establish that $\psi' < \psi$ for $t \in (0, T]$. Let $p = \psi/\psi'$ and note that

$$\frac{\dot{p}}{p} = \delta(p-1) + \delta\gamma(u-1). \tag{81}$$

Suppose, by contradiction, that $\psi'_0 > \psi$ so that $p_0 < 1$. Then Eq. (81) implies that p is falling for all t and p will never equal 1. But if $p > 1$ but $|\gamma(u-1)| > |(p-1)|$ then the second term in Eq. (81) will dominate the first, p will be falling over time and there will exist a T such that $p = 1$. $\square$

The only requirement that has been imposed on a switching equilibrium is that the switch must happen in finite time. By continuity, if there is an equilibrium for a given T, then there is a continuum of nearby switch dates all of which are equilibria as well. This result is similar to the indeterminacy of production locations in the case of the model with no externalities but it differs in one important respect. Since the alternative equilibria in the case of externalities entail different paths for the world production of physical commodities, they can be Pareto ranked.

6. Conclusion

The results reported in this paper represent a small part of a quantitative research program that aims to understand features of the international growth data using relatively simple models of endogenous growth. We have shown that a relatively standard two-country endogenous growth model will display multiple equilibria for very mild values of externalities. The set of equilibria in our model is relatively rich and it may involve symmetric equilibria in which countries follow balanced growth paths, asymmetric equilibria in which one country grows faster than another and combinations of these equilibria in which a country switches from low to high growth.

What should one conclude from the existence of equilibria of this kind? We do not suggest that our switching equilibria should be taken literally as a model of economic development; there are clearly many features of underdeveloped economies that we have not captured in this model. However, volatile movements of international capital and associated exchange rate crises are a prevalent feature of the data and, although we cannot address this issue in the non-stochastic model developed here, we conjecture that stochastic versions of our model may provide a useful vehicle for analyzing issues of this kind. We hope to pursue this idea in our future research.

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