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Indeterminacy with Non-separable Utility[1]

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J. Benhabib and R. E. A. Farmer (1994, *J. Econ. Theory* **63**, 19–41) showed that a single sector growth model in the presence of increasing returns-to-scale may display an indeterminate equilibrium if the demand and supply curves cross with the "wrong slopes." We generalize their result to a model with preferences that are non-separable in consumption and leisure. We provide a simple analog of the Benhabib–Farmer condition that works in the non-separable case. Our condition is easy to check in practice and it allows for equilibria to be indeterminate, even when demand and supply curves have the standard slopes. We illustrate that equilibrium can be indeterminate when demand and supply curves have standard slopes and the degree of increasing returns-to-scale is well within recent estimates by S. Basu and J. Fernald (1997, *J. Polit. Econ.* **105**, 249–283) for U. S. manufacturing. *Journal of Economic Literature* Classification Numbers: E10, E32, D90. © 2000 Academic Press

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1. INTRODUCTION

In this paper we study the conditions under which a single sector growth model with increasing returns-to-scale will display an indeterminate equilibrium. In a recent paper, Benhabib and Farmer [3] showed that the condition for equilibrium to be indeterminate in the one sector model with separable preferences is that the labor supply and demand curves should cross with the "wrong slopes." Our work generalizes their model to the case in which the preferences of the representative agent are non-separable in consumption and leisure.

The Benhabib–Farmer condition is intuitive and can be applied in practice to calibrate indeterminate models or to provide an econometric test of indeterminacy in a structural econometric model. But the assumption that utility is separable in consumption and leisure is restrictive since it implies that the intertemporal elasticity of substitution must equal one. Our generalization can be applied in practice to study the existence of indeterminacy in a much larger class of models. Arguably, the class that we study is the one most relevant to business cycle analysis with representative agent models since it is the largest class of growth models that is consistent with the stylized fact that hours worked in the U.S. have been stationary even though the real wage has grown.

Benhabib and Farmer defined the labor supply curve to be the quantity of labor supplied as a function of the real wage, holding constant consumption. When preferences are separable in consumption and leisure, the constant-consumption labor supply curve is identical to a second widely used concept; the Frisch labor supply curve, defined as the quantity of labor supplied as a function of the real wage holding constant the marginal utility of consumption.[4] When preferences are non-separable, the constant-consumption labor supply curve is different from the Frisch labor supply curve since holding constant the marginal utility of consumption is not the same as holding constant the level of consumption. We find that in a model with non-separable preferences, the appropriate condition for the indeterminacy of equilibrium is that the Frisch labor supply curve and the labor demand curve should cross with the "wrong slopes."

[4] Our usage of "Frisch" demand and supply functions follows Browning [6] and Browning *et al.* [7] who introduce the definition of a Frisch demand to refer to demands in which preferences are intertemporally separable and the demand functions for contemporaneous commodities are expressed as a function of current prices and of the Lagrange multiplier associated with an intertemporal budget constraint. Browning, Deaton, and Irish cite Frisch [12] as their source for the term.

2. RELATED LITERATURE

The Benhabib–Farmer condition for indeterminacy has been widely criticized as being implausible (see, for example, the discussion by Aiyagari [1]), since the required degree of returns-to-scale is higher than seems consistent with recent estimates. Basu and Fernald [2], for example, find that the returns-to-scale parameter in U.S. manufacturing is not much above unity. This observation has led a number of authors to study alternative approaches in which indeterminacy can be obtained more easily. Benhabib and Farmer [4] find that indeterminacy in multi-sector models does not require a high degree of increasing returns-to-scale, and Perli [17] is able to generate indeterminacy for reasonable parameter values in a model of home production. In a recent paper, Benhabib and Nishimura [5] show that indeterminacy can arise in a multi-sector model with constant returns-to-scale even when externalities are arbitrarily small. The multi-sector models studied by Benhabib and Farmer [4] and Benhabib and Nishimura [5] contain sector specific effects that are quite different from the aggregate externalities considered in our work.

Pelloni and Waldmann [16] generate indeterminacy in an endogenous growth model, with assumptions similar to ours. Their example is a limiting case of our model in which the social technology is linear in capital. In this case the equilibrium dynamics can be reduced to a first order difference equation in a single state variable. In our model, in contrast, the description of equilibrium dynamics requires two state variables, as in the standard real business cycle model. We show that, if one allows for non-separable preferences, equilibria may be indeterminate for returns-to-scale of 1.03. Our lower degree of returns-to-scale is in agreement with recent point estimates by Basu and Fernald and it allows for indeterminacy to occur even when the demand curve slopes down and the constant consumption supply curve slopes up. The Pelloni–Waldmann endogenous growth example is also consistent with standard sloped demand and supply curves, but it requires that returns-to-scale be equal to 1.66, when the labor elasticity is calibrated to equal labor's share of national income. This is unrealistically high.

Work by Lahiri [14] on indeterminacy in international models finds that open capital markets make indeterminacy more likely. Lahiri points out that open capital markets break the link between savings and investment and permit individuals to smooth consumption through international borrowing and lending, thereby making the representative agent behave in a more risk neutral manner. Our work on non-separabilities exploits a similar theme; the closer is σ to zero, the less averse is the consumer to fluctuations in consumption.

3. PRODUCTION TECHNOLOGY

Our technology is taken directly from Benhabib and Farmer [3]. We assume a large number of competitive firms, each of which produces a homogenous commodity using a constant returns-to-scale technology,

$$Y = K^a L^b A \tag{1}$$

where $a + b = 1$ and $A > 0$. Each firm takes the aggregate productivity shock A as given. However, A is determined in practice by the activity of other firms. We model externalities by the equation

$$A = \bar{K}^{\alpha - a} \bar{L}^{\beta - b}, \tag{2}$$

where $\bar{K}$ and $\bar{L}$ denote average economy wide use of capital and labor and, $1 > \alpha > a$, $\beta > b$, and $\alpha + \beta > 1$. We limit ourselves to the case when $\alpha < 1$, since when $\alpha = 1$ the dynamics of the model become one dimensional. This is the case of endogenous growth already studied by Pelloni and Waldmann [16]. In the case of $\alpha > 1$ growth is explosive and we rule this out by assumption. Substituting from (2) into (1) leads to an expression for the social production function,

$$Y = K^\alpha L^\beta. \tag{3}$$

We assume that factor markets are competitive and the factors of production receive fixed shares of national income,

$$b = \frac{wL}{Y}, \tag{4}$$

$$a = \frac{rK}{Y}, \tag{5}$$

where w is the wage rate, and r is the rental rate, both measured in terms of the consumption good. Factor shares in national income, a and b, will differ from the social marginal products, α and β, due to the existence of externalities in production.

4. THE CONSUMER'S PROBLEM

In this section, we describe the preferences of the representative consumer. We assume that consumers derive utility from the instantaneous utility function,

$$U(C, L) = \frac{[CV(L)]^{1-\sigma} - 1}{1 - \sigma}, \tag{6}$$

where $\sigma > 0$, $\sigma \neq 1$, and $V(L)$ is a non-negative, strictly decreasing concave function, bounded above, that maps $[0, \bar{L}] \to \Re$. We also assume that $V'(0)$ is bounded and $V(0) > 0$. $\bar{L}$ has the interpretation of the consumer's endowment of leisure and we allow for the possibility (since certain simple examples have this feature) that $\bar{L} = \infty$. The function $U(C, L)$ displays a constant intertemporal elasticity of substitution and generalizes the utility function used by Benhabib and Farmer [3],

$$U(C, L) = \ln C - \frac{L^{1+\gamma}}{1+\gamma}, \tag{7}$$

while maintaining the property that income and substitution effects exactly balance each other in the labor supply equation. This property is important since it captures the fact that labor supply per person is approximately stationary in the U.S. although the real wage has grown at an average rate of 1.6% per year in a century of data. When the consumer has unit elasticity of intertemporal substitution the parameter, σ, is one. In this case, if the function $V(L)$ is given by

$$V(L) = \exp\left(-\frac{L^{1+\gamma}}{1+\gamma} \right),$$

our utility function can be shown (using L'Hospital's Rule) to reduce to Eq. (7).

The representative consumer maximizes the present value of utility

$$\int_0^{\infty} U(C, L)\, e^{-\rho t}\, dt$$

subject to the budget constraint

$$\dot{K} = (r - \delta)\, K + wL - C,$$

the initial condition

$$K(0) = K_0,$$

and the "no Ponzi scheme" constraint

$$\int_{s=0}^{\infty} Q(s, 0)(C(s) - w(s)\, L(s)) \leqslant K(0),$$

where $\rho > 0$ is the discount rate, $0 < \delta < 1$ is the depreciation rate, and

$$Q(s,t) = \int_{v=t}^{s} e^{-r(v)+\delta}\,dv$$

is the price of a unit of consumption at date t for delivery at date s. Since the individual producers face constant returns-to-scale technologies, there are no profits in this economy.

5. SOLVING THE CONSUMER'S PROBLEM AND FINDING A MARKET EQUILIBRIUM

To solve the consumer's problem we define the present value Hamiltonian,

$$H = \frac{[CV(L)]^{1-\sigma} - 1}{1-\sigma} + \Lambda[(r-\delta)K + wL - C], \tag{8}$$

where Λ is the co-state variable.

The first order condition with respect to consumption is

$$\Lambda = C^{-\sigma} V(L)^{1-\sigma} \tag{9}$$

and with respect to labor supply is

$$w\Lambda = -C^{1-\sigma}\frac{V'(L)}{V(L)^{\sigma}}. \tag{10}$$

Substituting (9) into (10) and using the fact that the wage equals the marginal product of labor leads to the static condition

$$b\frac{Y}{L} = w = -C\frac{V'(L)}{V(L)}, \tag{11}$$

which is the requirement that the negative of the ratio of the marginal disutility of labor supply to the marginal utility of consumption should be equated to the real wage.

Along an optimal path the shadow value of capital, Λ, must obey the differential equation

$$\dot{\Lambda} = \Lambda\left(\rho + \delta - \frac{aY}{K}\right), \tag{12}$$

where we have used the firm's optimizing condition (5) in Eq. (12) to write the rental rate as a function of capital and labor. The transversality condition associated with this problem is represented by the equation

$$\lim_{t \to \infty} e^{-\rho t} \Lambda = 0. \tag{13}$$

To analyze the dynamics of a competitive equilibrium, the following transformations make the analysis easier. First, we divide the capital formation equation by the level of capital

$$\frac{\dot{K}}{K} = \frac{Y}{K} - \delta - \frac{C}{K}.$$

Defining lowercase letters, λ, l, k, c and y to be logarithms of their respective uppercase characters, the co-state equation becomes

$$\dot{\lambda} = \rho + \delta - a e^{y-k}, \tag{14}$$

and the capital accumulation equation is

$$\dot{k} = e^{y-k} - \delta - e^{c-k}. \tag{15}$$

We would like to analyze the stability of this pair of differential equations around a steady state. To do this, we must first find the steady state then obtain expressions for y and c in terms of the variables λ and k.

6. EXPRESSIONS FOR THE STEADY STATE

In this section we use the fact that

$$h(L) \equiv -\frac{V'(L)}{V(L)}$$

is monotonically increasing, to show that the model has a unique steady state. Monotonicity of $h(L)$ follows from the assumptions that $V'(L) > 0$ and $V''(L) < 0$ since

$$h'(L) = \frac{[V'(L)^2 - V(L)\, V''(L)]}{V(L)^2} > 0.$$

Notice also that since $V'(0)$ is bounded and $V(0) > 0$, $h(0)$ is positive and finite. We will use this property below to establish existence of a unique steady state value L^*.

We denote the logarithms of steady state variables $\{Y, C, K, L\}$ as $\{y^*, c^*, k^*, l^*\}$. To show uniqueness, first choose $\dot{\lambda}=0$, and solve Eq. (14) to find an expression for $y^* - k^*$,

$$y^* - k^* = \ln\left(\frac{\rho+\delta}{a}\right). \tag{16}$$

Similarly, setting $\dot{k}=0$, and using Eq. (16), solve Eq. (15) to give an expression for $c^* - k^*$,

$$c^* - k^* = \ln\left(\frac{\rho+\delta(1-a)}{a}\right). \tag{17}$$

It follows from (16) and (17) that $y^* - c^* = \ln(\frac{\rho+\delta}{\rho+\delta(1-a)}) > 0$ can be uniquely determined. From the labor market equation (11) we have

$$l^* + \ln(h(L^*)) = \log(b) + (y^* - c^*). \tag{18}$$

Let $f(L^*) \equiv \log(L^*) + \ln(h(L^*))$. Since $h(0)$ is finite $f(0) = -\infty$. Since h is increasing $f(L^*)$ is increasing and $f(L^*) \to \infty$ as $L^* \to \infty$. It follows that there is one and only one positive value of L^* for which (18) holds. Since L^* is bounded, there is some $\bar{L}$ for which $L^* \in [0, \bar{L}]$ and hence for any choice of utility function there is an upper bound on labor supply for which the equilibrium is interior. Given the value of L^* one can compute l^* and given the values of $y^* - k^*$ and $y^* - c^*$ one can use the production function (3) to solve for the individual variables y^*, k^*, and c^*.

7. DYNAMIC EQUILIBRIA

An equilibrium is a time path for the state variable k and the costate variable λ that satisfies the system

$$\dot{\lambda} = \rho + \delta - ae^{y-k} \tag{19}$$

$$\dot{k} = e^{y-k} - \delta - e^{c-k}, \tag{20}$$

with the boundary condition $k(0) = k_0$, and the transversality condition $\lim_{t\to\infty} e^{\lambda-\rho t} = 0$, together with a set of time paths for the variables c, l, and y that satisfy the side conditions

$$y = \alpha k + \beta l \tag{21}$$

$$c + \log(h(L)) = \log(b) + y - l, \tag{22}$$

$$\lambda = -\sigma c + (1-\sigma)\log(V(L)). \tag{23}$$

Equation (21) is the production function, (22) is the labor market first order condition and (23) is the first order condition for choice of consumption. Equations (21), (22), and (23) can be written as a set of approximate linear equations by defining the parameters

$$\psi = \left(\frac{-L^* V'(L^*)}{V(L^*)} \right) \tag{24}$$

$$\gamma = \frac{L^* h'(L^*)}{h(L^*)} \tag{25}$$

to yield the equations

$$\tilde{y} = \alpha \tilde{k} + \beta \tilde{l}, \tag{26}$$

$$(1+\gamma)\, \tilde{l} = \tilde{y} - \tilde{c} \tag{27}$$

$$\tilde{\lambda} = -\sigma \tilde{c} + \psi(\sigma - 1)\, \tilde{l}, \tag{28}$$

where tilde's denote deviations from the steady state.

The parameters ψ and γ have relatively simple interpretations. From Eq. (11) it follows that ψ is the share of wages relative to consumption since

$$\frac{-L^* V'(L^*)}{V(L^*)} = \frac{wL^*}{C}.$$

If we combine government and private consumption, the ratio of wage income to consumption has been approximately 1 since 1890 in U.S. data. It follows that ψ is approximately equal to 1. The parameter γ can also be recovered from data. If one linearizes (11) around the steady state, then γ would be the slope of the constant consumption labor supply curve.

8. LOCAL DYNAMICS

In this section, we analyze the local dynamics of the system around the unique steady state. We have described the economy by a pair of differential equations, in λ and k, (19) and (20) and by three static equations in the variables λ, l, y, and c. Notice first that the dynamic equations (19) and (20) are functions of $(y-k)$ and $(c-k)$. Our first task is to show that the three static Eqs. (26), (27), and (28) can be solved to find expressions for $(y-k)$ and $(c-k)$ as functions of λ and k. We find exact expressions for these variables in the appendix in which we derive Eq. (29) and find explicit parametric expressions for the elements of the matrix Φ,

$$\begin{bmatrix} \tilde{y} - \tilde{k} \\ \tilde{c} - \tilde{k} \end{bmatrix} = \Phi \begin{bmatrix} \tilde{\lambda} \\ \tilde{k} \end{bmatrix}. \tag{29}$$

Using the notation

$$\Phi \equiv \begin{bmatrix} \phi_1 & \phi_2 \\ \phi_3 & \phi_4 \end{bmatrix}$$

we can write the dynamics of this system around the steady state as an approximate linear system,

$$\begin{bmatrix} \dot{\lambda} \\ \dot{k} \end{bmatrix} = J \begin{bmatrix} \tilde{\lambda} \\ \tilde{k} \end{bmatrix}, \tag{30}$$

where we show in the appendix that the elements of J can be written as

$$J \equiv \begin{bmatrix} -a\phi_1\left(\dfrac{\rho+\delta}{a}\right) & -a\phi_2\left(\dfrac{\rho+\delta}{a}\right) \\ \phi_1\left(\dfrac{\rho+\delta}{a}\right) - \phi_3\left(\left[\dfrac{\rho+\delta}{a}\right] - \delta\right) & \phi_2\left(\dfrac{\rho+\delta}{a}\right) - \phi_4\left(\left[\dfrac{\rho+\delta}{a}\right] - \delta\right) \end{bmatrix}. \tag{31}$$

Since one variable of this pair of equations is predetermined, and the other is free, the system will have a locally unique (determinate) equilibrium when the steady state is a saddle; this requires one negative root and one positive root of the matrix J. If both roots of J are positive then paths that begin close to the steady state will move away from it. If these paths remain bounded then they will constitute valid equilibria. Alternatively, they may diverge and violate transversality and in general one cannot use local analysis to distinguish between the two possibilities. Indeterminacy of equilibrium occurs when both roots of J are negative. Since the trace of J is equal to the sum of the roots and the determinant is equal to their product, determinacy would require $\mathrm{Tr}(J) \lesseqgtr 0$, $\mathrm{Det}(J) < 0$, and indeterminacy that $\mathrm{Tr}(J) < 0$, $\mathrm{Det}(J) > 0$.

To characterize the conditions when indeterminacy occurs, as functions of the parameters of the model we establish the following two results:

PROPOSITION 1.

$$\mathrm{sign}(\mathrm{Det}(J)) = \mathrm{sign}(\eta),$$

where

$$\eta \equiv \sigma(\beta - \gamma - 1) - \psi(\sigma - 1).$$

PROPOSITION 2.

$$\mathrm{Tr}(J) = \rho + Q, \tag{32}$$

where

$$Q \equiv -\frac{(\rho+\delta)}{\eta}\left[(\sigma-1)\left(\beta - \psi\left(1 - \frac{\delta\alpha}{\rho+\delta}\right)\right) + d(1+\gamma)\,\sigma\right]$$

and

$$d \equiv \frac{(\alpha - a)}{a}.$$

Proofs of both results are given in the appendix. In the case when $\sigma = 1$, our model collapses to the Benhabib–Farmer model [3] and in this case η collapses to $\beta - 1 - \gamma$ and the determinant of J is positive when $\beta > 1 + \gamma$ as in Benhabib and Farmer. Notice also that $\mathrm{Tr}(J)$ is positive when there are no externalities and $\sigma = 1$ since, in this case, $\mathrm{Tr}(J) = \rho$. For small capital externalities, however, the trace of J becomes negative as soon as η passes through zero, from a small negative number to a small positive number since, when $\sigma = 1$,

$$Q = -\frac{(\rho+\delta)}{\eta}\, d(1+\gamma). \tag{33}$$

If η is small (close to zero) and positive then Q is large and negative and from proposition 2 it follows that the trace condition for indeterminacy is met. Hence, when $1 > \alpha > a$ and $\sigma = 1$, a necessary and sufficient condition for indeterminacy is that there exists a value η^* at which the trace of J switches sign. In the case of $\sigma = 1$ indeterminacy occurs when

$$0 < \eta < \eta^* = (\rho + \delta)\,\rho^{-1} d(1+\gamma).$$

The conditions for indeterminacy when σ is not equal to 1 are

(1) $d(1+\gamma)\,\sigma + (\sigma - 1)(\beta - \psi(1 - \frac{\delta\alpha}{\rho+\delta})) > 0$, and

(2) $\eta > 0$.

In this case indeterminacy occurs for values of η in the range

$$0<\eta<\eta^*=\frac{(\rho+\delta)}{\rho}\left[d(1+\gamma)\,\sigma+(\sigma-1)\left(\beta-\psi\left(1-\frac{\delta\alpha}{\rho+\delta}\right)\right)\right].$$

The reason for the condition that η should be positive is obvious since it implies that the determinant of J is positive. Condition 1 is sufficient to imply a negative trace at the point when η crosses 0 from above, since at this point η is small and positive and it follows from Eq. (33) that Q is large and negative, hence the trace of J is negative.

Condition 1 is satisfied when $\sigma=1$ for positive capital externalities ($d>0$) and, by continuity, for values of σ, close to one. In computational experiments we were able to generate examples of indeterminate equilibria for values of σ ranging from 0 to 2 although values of σ greater than 1 make it harder to generate an indeterminate equilibrium, since when $(\sigma-1)$ is strictly positive, β must be larger than would otherwise be the case for η to switch sign. In our calibrated examples we easily obtained indeterminacy for σ a little lower than 1 and β not much bigger than b. In our calibrations, $(\beta-\psi(1-\delta\alpha/(\rho+\delta)))$ was typically negative and so both terms of condition 1 were positive at the point where η changed sign. We show below that condition 2 is satisfied when the slopes of the labor demand curve and the Frisch labor supply curve cross with the "wrong slopes."

9. THE CASE OF ENDOGENOUS GROWTH

Our results on indeterminacy are related to the endogenous growth model of Pelloni and Waldmann [16] who study the case of a production function in which there is a capital externality, but no labor externality. The technology studied by Pelloni and Waldmann is

$$Y=F(\bar{K}L,\,K),$$

where K and L are private inputs of capital and labor and $\bar{K}$ is a capital externality. $F(X,\,Y)$ is constant returns-to-scale technology that is linearly homogenous in X and Y. When F is Cobb–Douglas, this structure is the limiting case of our model for $\alpha=1$ and $\beta=b$ (no labor externalities). Since Pelloni and Waldmann do not impose the assumption that F is Cobb–Douglas they are able to investigate the role of the elasticity of substitution between labor and capital in production on indeterminacy of the balanced growth path as well as considering the role of the elasticity of substitution of consumption and leisure in utility.

How does this model differ from ours? First, the equilibria of the Pelloni–Waldmann model are balanced growth paths that can be described by a difference equation in a single state variable. Benhabib and Farmer [3] in their original paper allowed for this case; we have ruled it out by assuming that $\alpha < 1$.[5] The endogenous growth version of the model will typically have multiple balanced growth paths, in contrast with our model in which the steady state equilibrium is unique. Pelloni and Waldmann consider the case with no labor externalities, and they are able to prove that indeterminacy occurs around any given balanced growth path when $\sigma < 1$ provided the production function is concave enough. "Concave enough" means that the production function in intensive form has a large negative second derivative and it is equivalent to the assumption that capital and labor are highly complimentary.[6]

Although the balanced growth version of the model is interesting, the magnitude of the capital externalities that are required for endogenous growth are extreme. If one calibrates the private production function using factor shares of 1/3 to capital and 2/3 to labor the aggregate technology in the Pelloni Waldmann version of the model would have increasing returns to the social production function of 5/3. We think that this is empirically implausible. The assumption that labor and consumption are non-separable is however, plausible, and there is considerable econometric evidence against the assumption of logarithmic utility over consumption. For this reason alone it is worth studying the model with small capital and labor externalities.[7] In related work, Perli [17] has shown that a model with home production can generate indeterminacy with a low degree of returns-to-scale and labor supply curves with "standard slopes." Perli's work is essentially a two-sector model in which one sector produces a non-marketed good. Our results are generated in the standard one-sector model and for this reason they are of independent interest.

[5] Generalizing our model to consider $\alpha = 1$ would lengthen our paper and add little.

[6] In our model, with labor externalities, indeterminacy can occur either for $\sigma < 1$ or for $\sigma \geqslant 1$ although it is still true that indeterminacy is more likely for the case of low σ. We restrict ourselves to a Cobb–Douglas technology because we hope to show that indeterminacy can arise in models that are calibrated in a way that can be compared directly with standard real business cycle economies, most of which uses a Cobb–Douglas production technology. The Pelloni–Waldmann results suggest that indeterminacy would be more likely if technology were calibrated as a CES production function with inputs that are compliments rather than substitutes.

[7] In related econometric work, Farmer and Ohanian [11] have estimated a structural model of the U.S. economy and used the results that we discuss in this paper to investigate the hypothesis that the U.S. economy is well described by a one sector model with an indeterminate balanced growth path. In contrast to the work by Farmer and Guo [10], Farmer and Ohanian [11] find evidence against indeterminacy. Their work relies in part on the generalized Benhabib–Farmer condition that we derive below.

10. THE LABOR MARKET AND INDETERMINACY

The indeterminacy condition of Benhabib and Farmer (that labor demand must slope up more steeply than labor supply) has been widely criticized as empirically implausible. (See, for example, the discussion by Aiyagari [1]). In this section, we show that this counter intuitive result is *not* necessary for indeterminacy in the more general case of non-separable preferences and we relate our conditions for indeterminacy to the slopes of labor demand and supply curves. Our main result is that the Benhabib–Farmer condition [3] that labor demand and supply curves cross with the "wrong slopes" generalizes to the non-separable case: but the correct concept of labor supply is the Frisch labor supply curve defined as labor supply as a function of the real wage holding constant the marginal utility of consumption.

10.1. *Separable Utility.* When $\sigma=1$ the utility function is logarithmic and the determinant of J is positive when

$$\beta-1>\gamma.$$

In this case the Frisch labor supply curve and the constant-consumption labor supply curve are identical and given by

$$\ln(w)=c+\gamma l,$$

and the labor demand curve is

$$\ln(w)=\text{constant}+\alpha k+(\beta-1)\,l.$$

Since the slope of the labor supply curve is γ and the slope of the labor demand curve is $\beta-1$, a necessary condition for indeterminacy is that the slope of the labor demand curve is larger than the slope of the labor supply curve.

10.2. *Non-separable Utility.* In the more general case when intertemporal substitution differs from one, the necessary condition for indeterminacy is that $\eta>0$ which implies, rearranging the definition of η, that

$$\beta-1>\frac{(\sigma-1)}{\sigma}\psi+\gamma.$$

In this case the Frisch labor supply curve and the constant-consumption labor supply curve differ. A linear approximation to the Frisch labor supply curve, in the neighborhood of the steady state, is given by

$$\ln(w)=\text{constant}-\frac{1}{\sigma}\lambda+\left(\frac{(\sigma-1)}{\sigma}\psi+\gamma\right)l. \tag{34}$$

If one substitutes for λ from Eq. (28) into Eq. (34) one obtains the constant-consumption labor supply curve

$$\ln(w) = \text{constant} + c + \gamma l, \tag{35}$$

which is identical (up to a constant) to the separable case. The labor demand curve is

$$\ln(w) = \text{constant} + \alpha k + (\beta - 1)\, l. \tag{36}$$

Notice that in general, the necessary condition for indeterminacy that $\eta > 0$, implies that the labor demand curve and the Frisch labor supply curve cross with the wrong slopes. Since the coefficient of l in the Frisch labor supply curve depends on the sign of $(\sigma - 1)$, indeterminacy may occur in the more general model when the labor demand curve slopes down. This may occur, for example, if $\sigma < 1$ and $\beta - 1$ (the slope of labor demand) is negative but greater than $(\psi(\sigma-1)/\sigma) + \gamma$ (the slope of Frisch labor supply). Note that in this case the Frisch labor supply curve would slope down also.

Equation (35) slopes up for all $\gamma > 0$ (a necessary condition for both consumption and leisure to be normal goods). It follows that, when the model is generalized to allow for differing degrees of intertemporal substitution, the labor demand curve and the constant-consumption labor supply curve may return to their traditional slopes even when the steady state is indeterminate.

11. A NUMERICAL EXAMPLE

In this section, we compare the dynamic properties of the model for alternative parameter values. We begin with a benchmark case given in Table I, in which the model has separable preferences and no externalities.

TABLE I

Benchmark Case

Parameter	Value	Description
ρ	0.065	Discount rate
σ	1	Coefficient of relative risk aversion
a	0.3	Capital share
m	1	Returns to scale
γ	0	Labor elasticity
δ	0.10	Depreciation rate

The returns-to-scale parameter, m is related to a and α by the equations

$$\alpha = am$$

$$\beta = bm.$$

In the benchmark case, the trace of the Jacobian matrix is ρ which is positive, and indeterminacy cannot occur. In Table II, in contrast, we vary the degree of returns to scale from 1 to 1.9. When m, reaches 1.43, there is a bifurcation in the system from a saddlepoint to a sink. At this point, the model displays an indeterminate steady state and is capable of generating business fluctuations driven purely by animal spirits as in the work of Farmer and Guo [9]. To obtain complex roots (Farmer and Guo argue that this is required to mimic the U.S. data) the returns to scale parameter must be increased still further to 1.48. Because recent empirical estimates (see for example the work by Basu and Fernald [2]) suggest that an upper bound on the degree of returns to scale in U.S. manufacturing is 1.09, the separable logarithmic case requires an implausibly high degree of returns-to-scale for the data to be consistent with indeterminacy.

In Table III, we look at a case where utility is slightly different from the separable logarithmic case; specifically we let the intertemporal substitution

TABLE II

Varies Returns to Scale, Benchmark Case

Returns to scale	Root 1	Root 2	Dynamics
1	0.45	−0.385	Saddlepath
1.1	−0.4017	0.5384	Saddlepath
1.2	−0.426	0.6972	Saddlepath
1.3	−0.4657	1.0807	Saddlepath
1.4	−0.55	3.915	Saddlepath
1.41	−0.5649	5.8338	Saddlepath
1.42	−0.5824	12.1974	Saddlepath
1.43	−0.6032	−70.2818	Sink
1.44	−0.629	−8.381	Sink
1.45	−0.6623	−4.2227	Sink
1.46	−0.7087	−2.6763	Sink
1.47	−0.7843	−1.8248	Sink
1.48	−1.0675 + 0.0848i	−1.0675 − 0.0848i	Sink
1.49	−0.9076 + 0.3621i	−0.9076 − 0.3621i	Sink
1.5	−0.7925 + 0.4344i	−0.7925 − 0.4344i	Sink
1.6	−0.38 + 0.4211i	−0.38 − 0.4211i	Sink
1.7	−0.2714 + 0.3432i	−0.2714 − 0.3432i	Sink
1.8	−0.2213 + 0.287i	−0.2213 − 0.287i	Sink
1.9	−0.1925 + 0.2443i	−0.1925 − 0.2443i	Sink

TABLE III

Varies Returns to Scale, Risk Aversion ($\sigma = 0.75$)

Returns to scale	Root 1	Root 2	Dynamics
1	-2.1306	2.1956	Saddlepath
1.01	-2.8655	3.0814	Saddlepath
1.02	-7.8212	107812	Saddlepath
1.03	$-0.2534+3.316i$	$-02535-3.316i$	Sink
1.04	$-0.1464+2.2681i$	$-0.1464-2.2681i$	Sink
1.05	$-0.1136+1.8294i$	$-0.1136-1.8294i$	Sink
1.06	$-0.0977+1.5729i$	$-0.0977-1.5729i$	Sink
1.07	$-0.0883+1.3885i$	$-0.0883-1.3995i$	Sink
1.08	$-0.0821+1.2721i$	$-0.0821-1.2721i$	Sink
1.09	$-0.0776+1.1733i$	$-0.0776-1.1733i$	Sink
1.1	$-0.0744+1.0938i$	$-0.0744-1.0938i$	Sink
1.2	$-0.0617+0.7087i$	$-0.0617-0.7087i$	Sink
1.3	$-0.0581+0.5532i$	$-0.0581-1.5532i$	Sink
1.4	$-0.0565+0.4619i$	$-0.0565-0.4619i$	Sink
1.5	$-0.0555+0.3992i$	$-0.0555-0.3992i$	Sink
1.6	$-0.0548+0.3522i$	$-0.0548-0.3522i$	Sink
1.7	$-0.0544+0.3148i$	$-0.0544-0.3148i$	Sink
1.8	$-0.054+0.2839i$	$-0.054-0.2839i$	Sink
1.9	$-0.0538+0.2575i$	$-0.0538-0.2575i$	Sink

parameter drop from 1 to 0.75. In this case we perform the same computational exercise and find that the system bifurcates from a saddlepoint to a sink at a much lower magnitude of returns to scale, 1.03. This is well within the empirically relevant range according to the estimates of Basu and Fernald. We conclude that by modifying the utility function to allow for varying degrees of intertemporal substitution, one is able to generate indeterminacy at a much lower magnitude of increasing returns than when the individual has logarithmic preferences.

12. DISCUSSION OF THE RESULTS

Although we have shown that indeterminacy may be consistent with a low degree of returns to scale this *does not* imply that the one sector model amended in this way can be used to generate business cycles when driven purely by sunspots in the manner described by Farmer and Guo [9]. When labor demand and constant consumption labor supply curves cross with the conventional slopes, purely sunspot driven business cycles will cause consumption and employment to move countercyclically; in the data they are procyclical. This is the same issue discussed by Benhabib and Farmer [4] in their two sector model. However, our model leads to the possibility that indeterminacy may provide an additional transmission mechanism for shocks originating in the real sector.

Our main result is that indeterminacy can arise in business cycle models that are very close to the standard real business cycle model if one allows utility to be non-separable in consumption and leisure. The early work in this area is generally thought to rely on an implausibly high degree of returns to scale. We have shown that a high degree of returns-to-scale is unnecessary if one is willing to specify preferences that are non-separable in consumption and leisure. The importance of our paper then hinges on the plausibility of our calibration of utility. On this point, we argue that the data is unclear and our parameterization is at least as plausible as the standard separable model in which the utility of consumption is logarithmic.

To generate indeterminacy with a low degree of returns-to-scale, we chose the curvature of the utility function to be on the linear side of logarithmic preferences (the parameter σ was chosen to be *smaller* than unity). In contrast, single sector models with a constant labor supply often include an assumption that the curvature parameter is *greater* than unity. Much of the literature on real business cycle models cites the econometric estimates of Hansen and Singleton [13] who estimate the coefficient of risk aversion, in a model with fixed labor supply, from the following moment condition

$$E_t\left[\left(\frac{C_{t+1}}{C_t}\right)^{-\sigma} R_{t+1}\right]=0. \tag{37}$$

In this equation C_{t+1}/C_t is consumption growth and R_{t+1} is the rate of return on an asset. Hansen and Singleton find a range of point estimates which place the coefficient of relative risk aversion (the parameter σ) "somewhere between 0 and 2."[8] On the surface, this would seem to be evidence in favor of our approach since 0.75 is at least as consistent with the Hansen and Singleton estimates as 2.0. However, the Hansen and Singleton estimates are not directly relevant to our model since our utility function is non-separable in consumption and leisure and it is not clear how one should map our more general example

$$U(C, L)=\frac{[CV(L)]^{1-\sigma}-1}{1-\sigma},$$

[8] Hansen and Singleton [13, p. 264].

into the function

$$U = \frac{C^{1-\sigma}}{1-\sigma}.$$

Mankiw Rotemberg and Summers [15] allow for a general class of non-separable utility functions and their estimates of the curvature parameters of utility, using U.S. quarterly data, fluctuate over a broad range of values as they vary the instrument list, the definition of consumption and the exact nature of the functional form. Although our functional is form is not nested within the class of utility functions used by Mankiw Rotemberg and Summers, their finding that parameter estimates are not robust to alternative specifications is unlikely to be altered by modifying their specification of utility. In related work that uses annual data, Farmer and Ohanian [11] estimate a complete equilibrium model using annual data from 1929 through 1999. Once again, standard errors are large and the data is unable to provide a precise estimate of the curvature parameter σ. We believe that the evidence is inconclusive because in the words of Cooley and Prescott, "... variations in the intertemporal elasticity of substitution affect transitions to balanced growth paths but not the paths themselves."[9] In the data, consumption varies very little but there are huge variations in the rate of return. In the absence of information from balanced growth paths of the kind that is used to estimate the elasticity of labor in the production function, the short run estimates will remain imprecise.

13. CONCLUSION

In this paper, we generalized the Benhabib–Farmer condition for indeterminacy to the case of non-separable preferences. Our condition is simple to check in practice and it covers a class of utility functions that is the most general class that is consistent with balanced growth. We found that, once one allows for non-separabilities between consumption and leisure, indeterminacy no longer requires that the demand curve and the constant consumption supply curve should cross with the wrong slopes. Instead, the required condition is that the labor demand curve and the *Frisch* labor supply curve should cross with non-standard slopes; a condition that is simple to check in practice. By means of an example, we showed that when the curvature parameter on the utility function is set at 0.75 in contrast to a value of unity that would hold in the logarithmic case, indeterminacy can occur at levels of increasing returns as low as 1.03.

[9] Cooley and Precott [8, p. 17].

14. APPENDIX

14.1. *Part* 1. *Deriving the Elements of Φ* . To derive the elements of the matrix Φ, we solve the static Eqs. (26), (27), and (28) for $(y-k)$ and $(c-k)$. We start by rearranging Eqs. (26) and (28) as a matrix system in the variables $(\tilde{y}-\tilde{k})$ and $(\tilde{c}-\tilde{k})$ which leads to the expression

$$\begin{bmatrix} 1 & 0 \\ 0 & \sigma \end{bmatrix}\begin{bmatrix} \tilde{y}-\tilde{k} \\ \tilde{c}-\tilde{k} \end{bmatrix}+\begin{bmatrix} -\beta \\ -\psi(\sigma-1) \end{bmatrix}\tilde{l}+\begin{bmatrix} 0 & 1-\alpha \\ 1 & \sigma \end{bmatrix}\begin{bmatrix} \tilde{\lambda} \\ \tilde{k} \end{bmatrix}=0. \tag{38}$$

We write the labor market equilibrium condition (27) separately in terms of the same linear combinations of variables,

$$\begin{bmatrix} -1 & 1 \end{bmatrix}\begin{bmatrix} \tilde{y}-\tilde{k} \\ \tilde{c}-\tilde{k} \end{bmatrix}+(1+\gamma)\,\tilde{l}=0. \tag{39}$$

Now, divide the second row of (38) by σ and divide Eq. (39) by $(1+\gamma)$ to obtain

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \tilde{y}-\tilde{k} \\ \tilde{c}-\tilde{k} \end{bmatrix}+\begin{bmatrix} -\beta \\ -\dfrac{(\sigma-1)}{\sigma}\psi \end{bmatrix}\tilde{l}+\begin{bmatrix} 0 & 1-\alpha \\ \dfrac{1}{\sigma} & 1 \end{bmatrix}\begin{bmatrix} \tilde{\lambda} \\ \tilde{k} \end{bmatrix}=0 \tag{40}$$

$$\begin{bmatrix} -\dfrac{1}{1+\gamma} & \dfrac{1}{1+\gamma} \end{bmatrix}\begin{bmatrix} \tilde{y}-\tilde{k} \\ \tilde{c}-\tilde{k} \end{bmatrix}+\tilde{l}=0. \tag{41}$$

Solve Eq. (41) for and substitute into Eq. (40) to obtain

$$\begin{bmatrix} \tilde{y}-\tilde{k} \\ \tilde{c}-\tilde{k} \end{bmatrix}+\begin{bmatrix} -\dfrac{\beta}{1+\gamma} & \dfrac{\beta}{1+\gamma} \\ \dfrac{-\psi(\sigma-1)}{\sigma(1+\gamma)} & \dfrac{\psi(\sigma-1)}{\sigma(1+\gamma)} \end{bmatrix}\begin{bmatrix} \tilde{y}-\tilde{k} \\ \tilde{c}-\tilde{k} \end{bmatrix}+\begin{bmatrix} 0 & 1-\alpha \\ \dfrac{1}{\sigma} & 1 \end{bmatrix}\begin{bmatrix} \tilde{\lambda} \\ \tilde{k} \end{bmatrix}=0.$$

Rearranging the above system one obtains

$$\begin{bmatrix} 1-\dfrac{\beta}{1+\gamma} & \dfrac{\beta}{1+\gamma} \\ \dfrac{-\psi(\sigma-1)}{\sigma(1+\gamma)} & 1+\dfrac{\psi(\sigma-1)}{\sigma(1+\gamma)} \end{bmatrix}\begin{bmatrix} \tilde{y}-\tilde{k} \\ \tilde{c}-\tilde{k} \end{bmatrix}+\begin{bmatrix} 0 & 1-\alpha \\ \dfrac{1}{\sigma} & 1 \end{bmatrix}\begin{bmatrix} \tilde{\lambda} \\ \tilde{k} \end{bmatrix}=0$$

or, equivalently,

$$A\begin{bmatrix}\tilde{y}-\tilde{k}\\ \tilde{c}-\tilde{k}\end{bmatrix}+B\begin{bmatrix}\tilde{\lambda}\\ \tilde{k}\end{bmatrix}=0,$$

where

$$A=\begin{bmatrix}1-\dfrac{\beta}{1+\gamma} & \dfrac{\beta}{1+\gamma}\\ \dfrac{-\psi(\sigma-1)}{\sigma(1+\gamma)} & 1+\dfrac{\psi(\sigma-1)}{\sigma(1+\gamma)}\end{bmatrix},\qquad(42)$$

$$B=\begin{bmatrix}0 & 1-\alpha\\ \dfrac{1}{\sigma} & 1\end{bmatrix}.$$

Solve for and $(\tilde{y}-\tilde{k})$ and $(\tilde{c}-\tilde{k})$ in terms of $\tilde{\lambda}$ and $\tilde{k}$,

$$\begin{bmatrix}\tilde{y}-\tilde{k}\\ \tilde{c}-\tilde{k}\end{bmatrix}=\Phi\begin{bmatrix}\tilde{\lambda}\\ \tilde{k}\end{bmatrix},$$

where

$$\Phi=-A^{-1}B=\begin{bmatrix}\phi_1 & \phi_2\\ \phi_3 & \phi_4\end{bmatrix}.$$

Inverting the expression for A, (42) it follows that the elements of A^{-1} are given by

$$A^{-1}=\frac{1}{\mathrm{Det}(A)}\begin{bmatrix}1+\dfrac{\psi(\sigma-1)}{\sigma(1+\gamma)} & \dfrac{-\beta}{1+\gamma}\\ \dfrac{\psi(\sigma-1)}{\sigma(1+\gamma)} & 1-\dfrac{\beta}{1+\gamma}\end{bmatrix},$$

where the determinant of A is

$$\mathrm{Det}(A)=\frac{\sigma(1+\gamma)+\psi(\sigma-1)-\beta\sigma}{\sigma(1+\gamma)}\equiv\frac{-\eta}{\sigma(1+\gamma)},\qquad(43)$$

and $\eta\equiv\sigma(\beta-\gamma-1)-\psi(\sigma-1)$. Note that the determinant of A is negative when η is positive.

The expression Φ is given by

$$\Phi = -A^{-1}B$$

$$= \frac{\sigma(1+\gamma)}{\eta}\begin{bmatrix} -\dfrac{\beta}{\sigma(1+\gamma)} & 1-\alpha+\dfrac{(1-\alpha)\,\psi(\sigma-1)}{\sigma(1+\gamma)}-\dfrac{\beta}{1+\gamma} \\ \dfrac{1+\gamma-\beta}{\sigma(1+\gamma)} & \dfrac{(1-\alpha)\,\psi(\sigma-1)}{\sigma(1+\gamma)}+1-\dfrac{\beta}{1+\gamma} \end{bmatrix}$$

the elements of which are

$$\phi_1 = -\frac{\beta}{\eta}, \tag{44}$$

$$\phi_2 = \frac{\sigma(1-\alpha)(1+\gamma)+(1-\alpha)\,\psi(\sigma-1)-\beta\sigma}{\eta}, \tag{45}$$

$$\phi_3 = \frac{1+\gamma-\beta}{\eta}, \tag{46}$$

$$\phi_4 = \frac{\sigma(1+\gamma)+(1-\alpha)(\sigma-1)\,\psi-\beta\sigma}{\eta}. \tag{47}$$

14.2. *Part* 2. *The Elements of J.* To find the elements of the matrix J, we use the definitions of ϕ_1, ϕ_2, ϕ_3, and ϕ_4, to write Eqs. (26) and (28) in the form

$$\tilde{y}-\tilde{k} = \phi_1\tilde{\lambda}+\phi_2\tilde{k}, \tag{48}$$

$$\tilde{c}-\tilde{k} = \phi_3\tilde{\lambda}+\phi_4\tilde{k}. \tag{49}$$

Now substitute these two equations into the two dynamic equations to obtain the expressions

$$\dot{\lambda} = \rho+\delta-ae^{\phi_1\tilde{\lambda}+\phi_2\tilde{k}+\theta_0}, \tag{50}$$

$$\dot{k} = e^{\phi_1\tilde{\lambda}+\phi_2\tilde{k}+\theta_0}-\delta-e^{\phi_3\tilde{\lambda}+\phi_4\tilde{k}+\theta_1}, \tag{51}$$

where $\theta_0 = y^*-k^*$ and $\theta_1 = c^*-k^*$ are constants that do not influence the dynamics. Linearizing Eqs. (50) and (51) leads to the system

$$\begin{bmatrix}\dot{\lambda}\\ \dot{k}\end{bmatrix} = J\begin{bmatrix}\tilde{\lambda}\\ \tilde{k}\end{bmatrix}, \tag{52}$$

where local information about the dynamics of the system is contained in the matrix J. The elements of J are given by the expression

$$J \equiv \begin{bmatrix} -ae^{\theta_0}\phi_1 & -ae^{\theta_0}\phi_2 \\ (\phi_1 e^{\theta_0} - \phi_3 e^{\theta_1}) & (\phi_2 e^{\theta_0} - \phi_4 e^{\theta_1}) \end{bmatrix}.$$

Using the steady state solutions of $\theta_0 = y^* - k^*$ and $\theta_1 = c^* - k^*$ from Eqs. (16) and (17) we can write this expression as

$$J \equiv \begin{bmatrix} -a\phi_1\left(\dfrac{\rho+\delta}{a}\right) & -a\phi_2\left(\dfrac{\rho+\delta}{a}\right) \\ \phi_1\left(\dfrac{\rho+\delta}{a}\right) - \phi_3\left(\left[\dfrac{\rho+\delta}{a}\right]-\delta\right) & \phi_2\left(\dfrac{\rho+\delta}{a}\right) - \phi_4\left(\left[\dfrac{\rho+\delta}{a}\right]-\delta\right) \end{bmatrix}$$

which is Eq. (31) in the text.

14.3. *Part* 3. *Proof of Proposition* 1. We prove, in this section, that the sign of the determinant of J depends on the sign of η, a variable that switches sign when the labor demand curve and the Frisch labor demand curves cross with the "wrong slopes." From Eq. (31) it follows that the determinant of J is given by the expression

$$\mathrm{Det}(J) = a(\phi_1\phi_4 - \phi_2\phi_3)\left(\frac{\rho+\delta}{a}\right)\left(\frac{\rho+\delta(1-a)}{a}\right). \tag{53}$$

The term, $(\phi_1\phi_4 - \phi_2\phi_3)$, is the determinant of Φ. Since a, $(\rho + \delta/a)$ and $(\rho + \delta(1-a))/a$ are all positive, the sign of the determinant of J is the same as the sign of the determinant of Φ. It follows that

$$\mathrm{sign}(\mathrm{Det}(J)) = \mathrm{sign}(\phi_1\phi_4 - \phi_2\phi_3) = \mathrm{sign}(\mathrm{Det}(\Phi)). \tag{54}$$

We now show that the determinant of Φ is related to the slopes of the labor demand and supply curves through the term η. Recall that Φ is defined as

$$\Phi = -A^{-1}B.$$

Using the properties of the determinant of a square matrix,

$$\mathrm{Det}(\Phi) = \mathrm{Det}(-A^{-1})\,\mathrm{Det}(B),$$

and since the matrix A is of dimension two,

$$\mathrm{Det}(\Phi) = (-1)^2\,\mathrm{Det}(A^{-1})\,\mathrm{Det}(B).$$

This implies that

$$\mathrm{Det}(\Phi) = \frac{\mathrm{Det}(B)}{\mathrm{Det}(A)}.$$

The determinant of B is

$$\mathrm{Det}(B) = \frac{\alpha - 1}{\sigma},$$

and it is negative in the view of the assumptions $0 < \alpha < 1$ and $\sigma > 0$. Therefore, the sign of the determinant of Φ is the opposite of the sign of the determinant of A, i.e.,

$$\mathrm{sign}(\mathrm{Det}(J)) = \mathrm{sign}(\mathrm{Det}(\Phi)) = -\mathrm{sign}(\mathrm{Det}\ A)).$$

But from definition of $\mathrm{Det}(A)$, Eq. (43) it follows that

$$\mathrm{sign}(\mathrm{Det}(A)) = -\mathrm{sign}(\eta).$$

Therefore the sign of the determinant of J is the same as the sign of η,

$$\mathrm{sign}(\mathrm{Det}(J)) = \mathrm{sign}(\eta).$$

Q.E.D

14.4. *Part* 4. *Proof of Proposition* 2. From the definition of the elements of J in Eq. (31) we can write the trace of J as

$$\mathrm{Tr}(J) = \left(\frac{\rho + \delta}{a}\right)(\phi_2 - \phi_4) + \phi_4 \delta - a\phi_1 \left(\frac{\rho + \delta}{a}\right). \tag{55}$$

Using Eqs. (45) and (47) note that

$$\phi_2 - \phi_4 = -\frac{\sigma\alpha(1+\gamma)}{\eta},$$

and

$$\phi_4 = -1 - \frac{\alpha\psi(\sigma - 1)}{\eta}.$$

Using these expressions we can rewrite (55) as

$$\mathrm{Tr}(J) = \underset{X}{\frac{(\rho+\delta)\,\beta}{\eta}} - \underset{Y}{\frac{(\rho+\delta)\,\sigma\alpha(1+\gamma)}{a\eta}} - \delta - \underset{Z}{\frac{\delta\alpha(\sigma-1)\,\psi}{\eta}}. \tag{56}$$

Now write the expressions X, Y, and Z, as

$$X = \frac{\beta}{\eta}(\rho+\delta)\,\sigma - (\sigma-1)(\rho+\delta)\,\frac{\beta}{\eta},$$

$$Y = -\frac{(\rho+\delta)\,\sigma(1+\gamma)}{\eta} - \frac{(\alpha-a)}{a}\,\frac{(\rho+\delta)\,\sigma(1+\gamma)}{\eta},$$

$$Z = -\frac{\psi(\sigma-1)(\rho+\delta)}{\eta} + \frac{\psi(\sigma-1)(\rho+\delta)}{\eta}\left(1-\frac{\delta\alpha}{\rho+\delta}\right).$$

Collecting together the first terms of each of the expressions for X, Y and Z and using the definition of $\eta = \sigma(\beta-\gamma-1) - \psi(\sigma-1)$ we can write the sum of the terms X, Y, and Z as

$$X + Y + Z = (\rho+\delta) + Q,$$

where

$$Q \equiv -\frac{(\rho+\delta)}{\eta}\left[(\sigma-1)\left(\beta-\psi\left(1-\frac{\delta\alpha}{\rho+\delta}\right)\right)+(1+\gamma)\,\sigma\right],$$

and

$$d \equiv \frac{(\alpha-a)}{a},$$

is a measure of the importance of capital externalities. Since $\mathrm{Tr}(J) = X + Y + Z - \delta$, we can write the trace of J as

$$\mathrm{Tr}(J) = \rho + Q$$

which is Eq. (32) in the text. Q.E.D.

REFERENCES

1. S. R. Aiyagari, The econometrics of indeterminacy: An applied study, a comment, *Carnegie–Rochester Conf. Ser. Public Policy* **43** (1995), 273–282.
2. S. Basu and J. Fernald, Returns to scale in U.S. production: Estimates and implications, *J. Polit. Econ.* **105** (1997), 249–283.
3. J. Benhabib and R. E. A. Farmer, Indeterminacy and increasing returns, *J. Econ. Theory* **63** (1994), 19–41.
4. J. Benhabib and R. E. A. Farmer, Indeterminacy and sector specific externalities, *J. Monet. Econ.* **37** (1996), 421–443.
5. J. Benhabib and K. Nishimura, Indeterminacy and sunspots with constant returns, *J. Econ. Theory* **81** (1998), 58–96.

6. M. J. Browning, "Profit Function Representations for Consumer Preferences," Bristol University Discussion Paper 82/125 1982.
7. M. J. Browning A. Deaton, M. Irish A profitable approach to labor supply and commodity demands over the life-cycle, *Econometrica* **53** (1985), 503–543.
8. T. F. Cooley and E. C. Prescott, Economic growth and business cycles, *in* "Frontiers of Business Cycle Research" (T. Cooley, Ed.), Princeton, NJ, 1995.
9. R. E. A. Farmer and J. T. Guo, Real business cycles and the animal spirits hypothesis, *J. Econ. Theory* **63** (1994), 42–72.
10. R. E. A. Farmer and J. T. Guo, The econometrics of indeterminacy: An applied study, *Carnegie–Rochester Conf. Ser. Public Policy* **43** (1995), 225–272.
11. R. E. A. Farmer and L. Ohanian, "The Preferences of the Representative American," Working Paper UCLA, March 1998.
12. R. Frisch, "New Methods of Measuring Marginal Utility," Tubige, J.C.B. Mohr, 1932.
13. L. P. Hansen and K. Singleton, Stochastic consumption, risk aversion and the temporal behavior of asset returns, *J. Polit. Econ.* **91** (1983), 249–265.
14. A. Lahiri, "Growth and Equilibrium Indeterminacy: The Role of Capital Mobility," Working Paper UCLA, January 1998.
15. N. G. Mankiw, J. R. Rotemberg, and L. H. Summers, Intertemporal substitution in macroeconomics, *Quart. J. Econ.* **100** (1985), 225–251.
16. A. Pelloni and R. Waldmann, Indeterminacy in a growth model with elastic labour supply, *Rivista Int. Sci. Sociali* **3** (1997), 202–211.
17. R. Perli, Indeterminacy, home production and the business cycle, *J. Monet. Econ.* **41** (1998), 105–125.