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Money in a Real Business Cycle Model

This paper constructs a real business cycle model in which real money balances yield utility. I calibrate the model to fit the first moments of U.S. data and I simulate a set of impulse response functions that are generated by the model for GDP, the rate of interest, money growth, and real balances. These theoretical impulse responses are compared with actual impulse responses from U.S. data. The model does a reasonably good job of capturing the dynamic interactions of money and real variables in U.S. data. It differs from most existing approaches by choosing a parameterization of utility for which the model admits the existence of indeterminate equilibria. I argue that this fact is critical in explaining the monetary propagation mechanism.

THIS PAPER IS ABOUT THE ROLE of money in initiating and sustaining business cycles. I am going to construct a model that is close to a real business cycle economy and I am going to use it to analyze some features of time series data, particularly the interaction between money, prices, output, and the rate of interest. The model is similar to work by other authors who have introduced money into an RBC economy. I have chosen to include a motive for holding money by including real balances as an argument of the utility function although this is not the main feature that distinguishes my work from related approaches.[1] My main innovation is to introduce a nonseparable specification of the utility function that allows me to make the case that money may be relatively important in financing transactions.

Previous authors have worked with models in which the motive for holding money is parameterized in a way that allows them to infer the properties of the demand function for money and the supply function of labor from first moments of the data. Since the share of social resources that is lost by using money is relatively small—these models imply that the marginal utility of money must be low. If the utility function for money is relatively simple, the properties of the second derivatives of utility (the parameters that determine the curvature of the function) can be directly inferred from the properties of the first moments of the data. It is these curvature

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1. Related work includes the calibrated cash in advance models of Cooley and Hansen (1989), or the liquidity effects model of Christiano and Eichenbaum (1992), Fuerst (1992), and Lucas (1990).

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parameters that determine the elasticities of the money demand function and the labor supply function.

Most monetary equilibrium models that have been analyzed in the literature contain a locally unique monetary equilibrium. Models with money do not necessarily display this property; in practice local uniqueness of the equilibrium is a consequence of the use of simple functional forms, such as logarithmic utility, that are calibrated to fit the fact that the interest component of GDP is small. The most important difference of my approach, from standard monetary equilibrium models, is that I include money in the utility function in a relatively flexible way. The underlying general equilibrium model that I will construct contains a continuum of equilibria; in the language of general equilibrium theory the steady-state equilibrium of the model is indeterminate.[2] The fact that the steady-state equilibrium is indeterminate has two important consequences for the properties of dynamic stochastic equilibria. First, it makes it particularly easy to construct examples of business cycles that are driven by shocks to the beliefs of the agents of the model. Second, and more importantly, the dynamic stochastic equilibria of the model display a rich internal propagation mechanism of a kind that mimics the dynamic characteristics of U.S. data.

In order to accept the indeterminacy approach that I lay out in this paper, one must accept that money plays a key role as a medium of exchange. One need not believe that this role is large in a direct sense; in fact, I will argue that the cost imposed on any individual trader is small. By this I mean that the percentage increase in utility gained by a 1 percent increase in consumption is approximately one hundred times greater than the percentage increase in utility gained by a 1 percent increase in real balances. But the fact that money is used by other traders in the economy means that there is a pecuniary externality in the use of money that is transmitted through the influence of the price level. If one trader holds additional nominal balances, the equilibrium price level must be correspondingly higher to maintain the same *real balances*. But the increase in price reduces the value of *other traders'* holdings of nominal money thereby causing them to react to the change in the first trader's decision. This effect is not present in standard general equilibrium analysis since the price of a commodity is not included as an argument of the utility function.[3]

Although I will model money as if it were equivalent to the monetary base, it is not necessary to assume that the medium of exchange is cash, or even checks. In the

2. The model that I will describe in this paper combines the real model of Benhabib and Farmer (1994) with the work on money in a production economy without capital (Benhabib and Farmer 1995). The current paper and the work reported in Benhabib and Farmer (1995) differ from the related work by Beaudry and Devereux (1993) who exploit increasing returns to scale to generate indeterminacy. Unlike Beaudry and Devereux and Benhabib and Farmer (1994), the model in this paper maintains a technology with constant returns to scale.

3. The idea in this paragraph, that the use of money yields an externality, is distinct from the alternative use of "externality" in Benhabib and Farmer (1993). Benhabib and Farmer argued that the real balances of one agent yield direct utility to another through an effect of the nominal holdings of one agent's money on the second agent's utility. In the current paper, externalities are transmitted through price and not through a direct effect of the nominal money stock.

paper, I report impulse response functions in data using M1 as a measure of money. But I use a very conservative estimate of the opportunity cost of holding money, less than one-fifth of 1 percent of output. In using this conservative estimate of the cost of holding money, I have in mind the fact that many firms and households carry out transactions using high-interest-yielding checking accounts that pay interest at a rate that is not much less than the yield on treasury securities. Indeed, many transactions between firms are carried out using trade credit that one might argue places *no* cost on the traders. In spite of my very low estimate of the opportunity cost of holding money, the model I will construct displays an indeterminate equilibrium. This would not have been the case had I modeled the utility of money using a simple separable utility function. Indeterminacy in my model arises from the fact that a more flexible functional form can capture the idea that the marginal utility of consumption and the disutility of effort are sensitive to the holdings of real balances. This does not seem to be an unreasonable assumption since the motive for including real balances in the utility function is to capture the idea that money may economize on the use of labor in transactions.

An important component of the mechanism that generates indeterminacy is that small increases in real balances must be associated with big increases, in equilibrium, of labor allocated to production.[4] In the paper, I do not distinguish between labor used in transactions and labor used in production. One could conceive of more disaggregated models that did make this distinction. In such a model one would not require that the absolute use of labor in transactions should be large. Instead, it must be the case that a small increase in real balances causes the household to supply a lot more labor to the market in an equilibrium. One way that this effect can be achieved is if the labor supply curve, as a function of the real wage, is very elastic. As the household carries additional real balances, the equilibrium real wage increases and causes the household to switch labor from the consumption of leisure to productive activity.

Why should one be attracted to models that use indeterminacy to explain monetary dynamics? Critics might point out that models with indeterminacy require one to give up on the main building block of the rational expectations assumption; that expectations are uniquely determined by fundamentals. Although this is true, it is not an unattractive feature. Equilibrium models with indeterminacy are not models in which "anything goes"; they place almost as many restrictions on the covariance properties of data as do standard rational expectations equilibrium models. It is true that the indeterminacy approach leaves some questions still to be addressed. How do individuals coordinate their expectations? Why is one equilibrium picked rather than another? But these questions are an order of magnitude less difficult than the obstacles posed by alternative explanations of the facts.

What are these alternatives? One widely used approach to modeling the monetary transmission mechanism is the nominal contracting model of Taylor (1979). This

4. I elaborate on this idea in Farmer (1997).

model is troublesome because Taylor contracting models rely on a nominal rigidity in the wage contract that remains unexplained. Furthermore, agents in a Taylor model are forbidden to write contracts indexed to nominal magnitudes that could potentially increase their welfare. Even a small reduction in the restrictions on indexation or on the rationality of the traders leads to a model that is no longer able to explain persistence of monetary shocks.[5] A second alternative that goes some way toward capturing the impact effects of monetary shocks is the financial constraint model of Christiano and Eichenbaum (1992) in which agents are prevented from carrying out their transactions in an order that they would prefer. But although restrictions on trade of this kind can capture the impact effects of monetary shocks, they too are unable to capture the persistence of monetary shocks that is one of the dominant characteristics of the impulse responses found in data. A third alternative is the menu cost approach of Akerlof and Yellen (1985) and Mankiw (1985). Once again, menu costs can capture the impact effects of monetary policy but they do not do a good job of explaining monetary dynamics.

In summary, the three mainstream approaches to explaining the monetary transmission mechanism are nominal contracting, financial constraints, and menu costs. These three approaches all suffer from the same two defects. First, they do not motivate the environment of the model in a way that explains *why* agents would choose to act in the ways that they do. Second, they cannot generate endogenous persistence of shocks. A common feature of all three approaches is that they explain data with a model in which there exists a unique determinate rational expectations equilibrium. In contrast to these approaches I am advocating the use of standard general equilibrium theory amended to include real balances as an argument of the utility function in a way that was first suggested by Patinkin nearly fifty years ago. By parameterizing this model in a way that is consistent with simple monetary statistics from U.S. data I will show that the parameterized model has an indeterminate set of equilibria that are indexed by beliefs. Using the calibrated model I will show that one is able to generate simulated monetary business cycles and that impulse response functions computed from artificial data look a lot like the impulse response functions from annual U.S. time series. Although the impulse response functions that I will report occasionally stray outside of the standard error bounds of the data, they are several orders of magnitude closer to the actual data than the monetary dynamics implied by any of the three leading alternatives.

1. THE MODEL

I assume that the economy consists of a large number of representative families each of which maximizes the present discounted value of a lifetime utility function.

5. Chari, Kehoe, and McGrattan (1996) modify the Taylor model by deriving the weights in the price-setting equation endogenously. They show that there are no preferences chosen from a standard class that can explain persistence.

$$\text{Max } \tilde{U} = E_1 \left\{ \sum_{t=1}^{\infty} \beta^{t-1} U\left(C_t, \frac{M_t}{P_t}, L_t\right) \right\} \tag{1}$$

where C_t is consumption, L_t is labor supply, and M_t/P_t is the real value of money balances. The period budget constraint of each family is given by

$$\begin{aligned} M_t + B_t + P_t K_{t+1} &= M_{t-1} + B_{t-1}(1 + i_{t-1}) + P_t K_t (1 - d) \\ &\quad + (Y_t - C_t + T_t)P_t , \quad t = 1, \ldots \infty , \end{aligned} \tag{2}$$

where M_t is the nominal stock of money and B_t is the nominal stock of government one-period bonds held from period t to $t + 1$. P_t is the price of commodities in terms of money and i_t is the rate of interest on bonds. K_t is the stock of capital, d is the rate of depreciation, Y_t is GDP, and T_t is a real transfer received by the representative family from the government at the beginning of period t. The household also faces a resource constraint described by equation (3),

$$K_{t+1} = K_t(1 - d) + Y_t - C_t , \tag{3}$$

and a debt limit, inequality (4), that prevents the household from borrowing forever and never paying back its debt.

$$\lim_{t \to \infty} Q_s^t \left(\frac{M_t + B_t}{P_t} + K_{t+1} \right) \geq 0, \qquad Q_s^t = \frac{P_t}{P_s} \frac{1}{\prod_{j=s}^{t-1} (1 + i_s)} . \tag{4}$$

The variable Q_s^t in this inequality is the date s value of a period t dollar.

To describe production, I assume that output is produced from labor and capital using the technology:

$$Y_t = (A_t L_t)^{\alpha} K_t^{1-\alpha} V_t , \tag{5}$$

where A_t is a nonstochastic trend and V_t is a random productivity shock. I will model growth as a stationary process by making the following assumptions about the properties of A_t and V_t:

$$A_t = \gamma^t A_0 , \tag{6}$$

$$V_t = V_{t-1}^{\zeta} \exp(u_t^1) . \tag{7}$$

The parameter γ is the growth factor of per capita GDP, ζ measures the persistence of the technology shock, and u_t^1 is its innovation. I assume that u_t^1 has zero expected value, is independently and identically distributed (i.i.d.) through time, and has small bounded support. The assumption of small bounded support is required to ensure that a linear approximation to the equilibrium of the model will remain close to the actual equilibrium. Later in the paper I will introduce an additional fundamental shock, u_t^2, that represents a policy disturbance and I will introduce two sunspot shocks that I will refer to as e_t^1 and e_t^2. All three of these additional shocks will also be assumed to be i.i.d. stochastic processes with small bounded support although I will allow for the possibility that they may be contemporaneously correlated.

2. MODELING PREFERENCES

In this section I discuss the class of utility functions that I will use in my quantitative analysis. The choice of functional form is important since it will play a key role in allowing me to solve for linear approximations to the equilibria of the model and for allowing these equilibria to mimic both low-frequency and high-frequency features of the data. My choice of functional form was guided by two considerations: (i) the fact that consumption, investment, and GDP have common trends (growth is balanced) and (ii) the velocity of circulation and the rate of interest have a common trend.[6] These facts require that there exist a representation of the equations of the model in terms of transformed variables that are independent of time. The class of utility functions that I will introduce below, when combined with a technology that displays labor augmenting technical progress such as the Cobb-Douglas technology in equation (5), satisfies this condition.

It is well known[7] in the case of the nonmonetary growth model that the period utility function must be of the form, $U(C, L) = (C^{1-\rho}/1 - \rho V)(L)$ in order for the model to admit the existence of a balanced growth path.[8] In the case of a monetary model, the restriction to utility functions that admit the possibility of balanced growth implies that the period utility function $U(C, M/P, L)$ should be homogeneous in C and M/P. This homogeneity requirement places a number of restrictions on the class of admissible utility functions that are discussed in more depth below.

3. RESTRICTIONS ON THE PARAMETER SPACE

Since I will be concerned with a linearized version of the model, it will be helpful to begin by defining its parameters in terms of the elasticities of the utility function and of its partial derivatives. These parameters, defined in equations (8)–(12), to-

6. King, Plosser, Stock and Watson (1991).

7. King, Plosser, and Rebelo (1988).

8. In the postwar annual data, there is some evidence that employment has been trending up and hence, a model that displays a balanced growth path in which consumption, capital, and GDP grow at the same rate and in which employment is stationary may not be the best description of the data. Nevertheless, I have chosen to begin an exploration of a monetary model by making a minimal departure from the existing literature and for this reason I will use a utility function that will reduce to equation (8) in the absence of money.

gether with the parameters of the production function, completely characterize the behavior of the linearized model around its balanced growth path:

$$U_c = \frac{\partial U}{\partial C}, \qquad U_m = \frac{\partial U}{\partial (M/P)}, \qquad U_L = -\frac{\partial U}{\partial L}, \tag{8}$$

$$\delta_c = \frac{C}{U} U_c, \qquad \delta_m = \frac{(M/P)}{U} U_m, \qquad \delta_L = \frac{L}{U} U_L. \tag{9}$$

$$\delta_{cc} = \frac{C}{U_c} \frac{\partial U_c}{\partial C}, \qquad \delta_{cm} = \frac{(M/P)}{U_c} \frac{\partial U_c}{\partial (M/P)}, \qquad \delta_{cL} = \frac{L}{U_c} \frac{\partial U_c}{\partial L}, \tag{10}$$

$$\delta_{mc} = \frac{C}{U_m} \frac{\partial U_m}{\partial C}, \qquad \delta_{mm} = \frac{(M/P)}{U_m} \frac{\partial U_m}{\partial (M/P)}, \qquad \delta_{mL} = \frac{L}{U_m} \frac{\partial U_m}{\partial L}, \tag{11}$$

$$\delta_{Lc} = \frac{C}{U_L} \frac{\partial U_L}{\partial C}, \qquad \delta_{Lm} = \frac{(M/P)}{U_L} \frac{\partial U_L}{\partial (M/P)}, \qquad \delta_{LL} = \frac{L}{U_L} \frac{\partial U_L}{\partial L}. \tag{12}$$

Notice that U_L is defined as the *negative* of the marginal disutility of labor supply. In matrix form one can describe the parameters of the utility function as a vector of three elements, δ:

$$\delta = \{\delta_c, \delta_m, \delta_L\}, \tag{13}$$

and a matrix D:

$$D = \begin{bmatrix} \delta_{cc} & \delta_{cm} & \delta_{cL} \\ \delta_{mc} & \delta_{mm} & \delta_{mL} \\ \delta_{Lc} & \delta_{Lm} & \delta_{LL} \end{bmatrix} \tag{14}$$

Although D has nine elements, only six of them are unrestricted since D is related to the Hessian matrix of the utility function which is symmetric. The symmetry of the Hessian imposes the following three restrictions on D:

$$\delta_{cm} = \delta_{mc} \frac{\delta_m}{\delta_c}, \qquad \delta_{cL} = -\delta_{Lc} \frac{\delta_L}{\delta_c}, \qquad \delta_{mL} = -\delta_{Lm} \frac{\delta_L}{\delta_m}. \tag{15}$$

The fact that the data displays (approximately) balanced growth places an additional three restrictions on the parameter space that are dictated by the requirement that the utility function should be homogeneous. If we let utility be homogeneous of degree $(1 - \rho)$ in C and M/P, the set of free parameters can be described by $\{\rho, \delta, D\}$ together with the symmetry restrictions, (15), and the homogeneity restrictions:

$$\delta_c + \delta_m = 1 - \rho, \qquad \delta_{cc} + \delta_{cm} = -\rho, \qquad \delta_{mc} + \delta_{mm} = -\rho,$$

$$\delta_{Lc} + \delta_{Lm} = 1 - \rho. \tag{16}$$

Imposing these restrictions directly on D, one has a set of parameters described by the vector δ and the elements of the matrix D restricted in the way described in equation (18):

$$\delta = \{\delta_c, \delta_m, \delta_L\} \tag{17}$$

$$D = \begin{bmatrix} -\rho - \delta_{cm} & \delta_{cm} & \delta_{cL} \\ \delta_{cm}\dfrac{\delta_c}{\delta_m} & -\rho - \delta_{cm}\dfrac{\delta_c}{\delta_m} & -\left(1 - \rho + \delta_{cL}\dfrac{\delta_c}{\delta_L}\right)\dfrac{\delta_L}{\delta_m} \\ -\delta_{cL}\dfrac{\delta c}{\delta_L} & \left(1 - \rho + \delta_{cL}\dfrac{\delta_c}{\delta_L}\right)\dfrac{\delta_L}{\delta_m} & \delta_{LL} \end{bmatrix}. \tag{18}$$

The complete class of preferences that is consistent with utility maximization and balanced growth allows for relatively flexible functional forms. In this paper I will exploit the flexibility of the utility function to parameterize the model in a way that is consistent with the existence of an indeterminate equilibrium. It is the fact that the equilibrium of the model is indeterminate that allows me to construct equlibria that mimic the propagation mechanisms for business cycles that we observe in time series data.

4. A PARAMETRIC UTILITY FUNCTION

In the quantitative section of this paper I will calibrate some of the parameters of the utility function to capture the first moments of the data. If one restricts oneself to simple logarithmic or CES utility functions, the calibration of the first moments also restricts the elements of D in such a way that the behavior of the equilibria of a parameterized model may be relatively uninteresting because the choice of the elements of δ completely determines the elements of D. The following class of functions allows for a much richer variety of possible equilibrium behaviors:

$$U\left(C, \frac{M}{P}, L\right) = \frac{X\left(C, \frac{M}{P}\right)^{1-\rho}}{1-\rho} - W\left(C, \frac{M}{P}\right)^{1-\rho} V(L), \quad \rho > 1 \tag{19}$$

where

$$V(L) = L^{1+\chi}, \qquad \chi > 0, \tag{20}$$

$$X\left(C, \frac{M}{P}\right) = \left[(1-a)C^{1-\lambda} + a\left(\frac{M}{P}\right)^{1-\lambda}\right]^{1/1-\lambda}, \qquad \lambda > 0, \lambda \neq 1 \tag{21}$$

$$W\left(C, \frac{M}{P}\right) = \left[(1-b)C^{1-\lambda} + b\left(\frac{M}{P}\right)^{1-\lambda}\right]^{1/1-\lambda}, \qquad \lambda > 0, \lambda \neq 1 \tag{22}$$

The utility function in (19) is a weighted sum of CES aggregators in which each individual CES aggregator combines consumption and real balances; the weight between the two functions W and X is determined by the use of labor as described by the disutility of effort, $V(L)$. In my quantitative analysis I will restrict attention to the class of utility functions in which the homogeneity parameter, ρ, is strictly greater than unity mainly because in versions of this utility function in which money plays no role the parameter ρ would have the interpretation of the "coefficient of relative risk aversion"; there is a consensus in much of the literature that this parameter should be in the range of 1 to 4. The parameter λ is the elasticity of substitution in each of the two aggregator functions W and X. I have restricted this parameter to be the same in each case simply because I am able to solve this special case for the balanced growth path and the ability to solve the model analytically was important in suggesting where in the parameter space, to search for a functional form that can mimic the dynamic responses of data. The parameter λ plays an important role since it allows me to calibrate the model in a way that captures the interest coefficient that one would expect to observe in estimates of the "demand for money" that would be obtained from data generated by the model. The parameters a and b represent the relative importance of money and they play an important role in allowing the model to capture the fact that the interest cost of holding money is small in observed data. By making λ relatively large but a and b very small, one can capture the fact that the "direct effect" of money is small but its "indirect effect" is big. By the direct effect, I mean the marginal utility of money evaluated at the steady state and by the indirect effect I mean the cross partial of money with consumption. The idea that one captures with this added flexibility in the utility function is that additional units of real balances do not yield much utility in themselves but they may nevertheless be highly complementary with other commodities and with labor supply.

5. THE SOLUTION TO THE INDIVIDUAL PROBLEM

In this section I will describe the solution to the household's optimizing problem. The household chooses sequences of state-contingent money, debt, capital, and labor supply to maximize (1) subject to the constraints (2), (3), and (4) and the initial conditions:

$$M_0 = \bar{M}_0, \qquad B_0 = \bar{B}_0, \qquad K_1 = \bar{K}\,. \tag{23}$$

The solution to this problem is given by the first-order conditions:

Labor,

$$\frac{U_L\left(C_t, \dfrac{M_t}{P_t}, L_t\right)}{U_c\left(C_t, \dfrac{M_t}{P_t}, L_t\right)} = \frac{\alpha Y_t}{L_t}\,, \tag{24}$$

Money,

$$U_c\left(C_t, \frac{M_t}{P_t}, L_t\right)\left[1 - \frac{U_m\left(C_t, \frac{M_t}{P_t}, L_t\right)}{U_c\left(C_t, \frac{M_t}{P_t}, L_t\right)}\right]$$

$$= E_t\left\{\beta U_c\left(C_{t+1}, \frac{M_{t+1}}{P_{t+1}}, L_{t+1}\right)\frac{P_t}{P_{t+1}}\right\}, \tag{25}$$

Bonds,

$$U_c\left(C_t, \frac{M_t}{P_t}, L_t\right) = E_t\left\{\beta U_c\left(C_{t+1}, \frac{M_{t+1}}{P_{t+1}}, L_{t+1}\right)\frac{P_t}{P_{t+1}}(1 + i_t)\right\}, \tag{26}$$

Capital,

$$U_c\left(C_t, \frac{M_t}{P_t}, L_t\right) = E_t\left\{\beta U_c\left(C_{t+1}, \frac{M_{t+1}}{P_{t+1}}, L_{t+1}\right)\right.$$

$$\left.\left[1 - d + \frac{(1-\alpha)Y_{t+1}}{K_{t+1}}\right]\right\}. \tag{27}$$

Equation (24) is the static first-order condition for choice of L_t, and equations (25), (26), and (27) are the dynamic first-order conditions for choice of M_t, B_t, and K_{t+1}.

6. GOVERNMENT POLICY

Since my main focus in this paper will be the conduct of monetary policy, I am going to make a particularly simple assumption about the conduct of fiscal policy; I will model fiscal policy with the assumption:

$$B_t = 0, \qquad \text{for all } t. \tag{28}$$

To model monetary policy I assume that government follows the feedback rule:

$$\bar{I}(x_t, x_{t-1}, u_t^2) = 0 \tag{29}$$

where x is a vector of the endogenous variables of the model, u^2 is a policy shock, and $\bar{I}$ is a function that represents the reaction of the Fed to current and past variables in the economy. A particularly simple policy is given by the fixed interest rate rule:

$$I_t = \frac{i_t}{1 + i_t} = \bar{I}. \tag{30}$$

I will use the rule in (30) in my subsequent analysis to evaluate the stability of the model since the fixed interest rate rule leads to a model with relatively simple dynamics. Equation (29) is, however, much more general. For example, the fixed money growth rate suggested by Friedman can be fitted into this framework, as can most of the stabilization rules suggested in recent literature, by adjusting the function $I()$.[9] In the analysis in the paper I will linearize (29) in the neighborhood of a fixed interest rate and in this linear analysis the choice of different monetary rules can be analyzed as the choice of the coefficients of a linear equation.

Given the fiscal rule, (28), the budget equation (2), and the capital accumulation definition (3), the transfer variable, T_t, is defined by the equation:

$$T_t = \frac{M_t - M_{t-1}}{P_t} . \tag{31}$$

7. EQUILIBRIA

Using the policy rule and market-clearing assumptons one can define an equilibrium as a set of stochastic processes for the variables, $\{C, K, V, M, P, Y, L\}$ that obey the following equations:

Capital accumulation,

$$K_{t+1} = K_t(1 - d) + Y_t - C_t , \tag{32}$$

Production function,

$$Y_t = (A_t L_t)^{\alpha} K_t^{1-\alpha} V_t , \tag{33}$$

Productivity shock,

$$V_t = V_{t-1}^{\zeta} \exp(u_t^1) , \tag{34}$$

Labor market equilibrium,

$$\frac{U_L\left(C_t, \dfrac{M_t}{P_t}, L_t\right)}{U_c\left(C_t, \dfrac{M_t}{P_t}, L_t\right)} = \alpha \frac{Y_t}{L_t} , \tag{35}$$

Bond Euler equation,

$$U_c\left(C_t \frac{M_t}{P_t}, L_t\right)(1 - I_t) = \beta E_t\left\{ U_c\left(C_{t+1} \frac{M_{t+1}}{P_{t+1}}, L_{t+1}\right) \frac{P_t}{P_{t+1}} \right\} , \tag{36}$$

9. For an analysis of the effect on the equilibrium of alternative monetary policies in a model that is closely related to this one, see Bennett (1996).

Capital Euler equation,

$$U_c\left(C_t \frac{M_t}{P_t}, L_t\right) = \beta E_t\left\{U_c\left(C_{t+1} \frac{M_{t+1}}{P_{t+1}}, L_{t+1}\right)\left[1 - d + (1-\alpha)\frac{Y_{t+1}}{K_{t+1}}\right]\right\}, \quad (37)$$

Monetary equilibrium,

$$\frac{U_m\left(C_t, \frac{M_t}{P_t}, L_t\right)}{U_c\left(C_t, \frac{M_t}{P_t}, L_t\right)} = I_t . \quad (38)$$

Equation (38) combines the Euler equations for money and bonds (25) and (26). An equilibrium also requires that the sequences $\{C, K, Y, M/P\}$ should not grow "too fast" in order to satisfy a transversality condition that is necessary for a candidate sequence to be the solution to an individual's problem. All of the equilibria that I will study will be described as steady states in transformed variables and one may show that boundedness of these transformed variables is sufficient to guarantee that the transversality condition holds.

8. TRANSFORMED VARIABLES

Sections 8 and 9 of the paper show how to reduce the equations of the model to a more manageable system that can be handled by linear methods. A necessary condition for the use of linear methods is that the model should display a fixed point around which linearization makes sense. It is partly for this reason that I have chosen functional forms for the production function and for the utility function that are consistent with the existence of a balanced growth path. To illustrate the existence of a balanced growth path, I will demonstrate that the equations of the model can be rewritten in terms of a set of *transformed* variables each of which is independent of time. Since the transformed equations are autonomous, it is possible to search for the existence of a steady-state solution to these equations. If such a solution exists, it will define a balanced growth path.

The following equations define the transformed variables that I will use in this study:

$$y_t = \frac{Y_t}{A_t}, \quad k_t = \frac{K_t}{A_t}, \quad c_t = \frac{C_t}{A_t}, \quad m_t = \frac{M_{t-1}}{P_t A_t}, \quad \mu_t = \frac{M_t}{M_{t-1}}, \quad \gamma = \frac{A_t}{A_{t-1}} . \quad (39)$$

Lowercase y, k, and c are the ratios of consumption, GDP, and capital to the productivity trend, A_t. The variable, lowercase m, is the ratio of real balances to the productivity trend where real balances are defined relative to last period's nominal stock of money. This choice of state variable will be convenient for a study of monetary dynamics since it will enable me to monitor innovations to the price level independently of innovations to the nominal stock of money. I will keep track of innovations to the stock of money with the variable μ (the money growth factor) and innovations to the price level with the variable m. An innovation to the price level will cause m to fall and an innovation to the money stock will cause μ to rise.

Using the definitions in equations (39) one may rewrite the equations of the model:

Production function,

$$y_t = (L_t)^{\alpha}(k_t)^{1-\alpha}V_t , \tag{40}$$

Labor market equilibrium,

$$c_t \frac{U_L\left(1, \frac{\mu_t m_t}{c_t}, L_t\right)}{U_c\left(1, \frac{\mu_t m_t}{c_t}, L_t\right)} = \alpha \frac{Y_t}{L_t} , \tag{41}$$

Monetary equilibrium,

$$\frac{U_m\left(1, \frac{\mu_t m_t}{c_t}, L_t\right)}{U_c\left(1, \frac{\mu_t m_t}{c_t}, L_t\right)} = I_t , \tag{42}$$

Capital accumulation,

$$\gamma k_{t+1} = k_t(1 - d) + y_t - c_t , \tag{43}$$

Bond Euler equation,

$$c_t^{-\rho} U_c\left(1, \frac{\mu_t m_t}{c_t}, L_t\right)(1 - I_t) = \frac{\beta}{\gamma^{-\rho}} E_t \cdot \left\{ c_{t+1}^{-\rho} U_c\left(1, \frac{\mu_{t+1} m_{t+1}}{c_{t+1}}, L_{t+1}\right) \frac{m_{t+1}}{m_t} \frac{1}{\mu_t} \right\} , \tag{44}$$

Capital Euler equation,

$$c_t^{-\rho} U_c\left(1, \frac{\mu_t m_t}{c_t}, L_t\right) = \frac{\beta}{\gamma^{-\rho}} E_t \cdot \left\{ c_{t+1}^{-\rho} U_c\left(1, \frac{\mu_{t+1} m_{t+1}}{c_{t+1}}, L_{t+1}\right)\left[1 - d + (1 - \alpha) \frac{y_{t+1}}{k_{t+1}}\right]\right\} , \tag{45}$$

Productivity shock,

$$V_t = V_{t-1}^{\zeta}\exp(u_1^t) \,, \tag{46}$$

Policy rule,

$$I_t = \bar{I}(x_t, x_{t-1}, u_t^2) \,. \tag{47}$$

Equations (41), (42), (44), and (45) exploit the fact that U_L is homogeneous of degree $1 - \rho$ and U_c and U_m are homogenous of degree ρ.

9. BALANCED GROWTH

In this section I am going to search for a point around which to linearize equations (40)–(47). A natural candidate for linearization is the balanced growth path of a nonstochastic version of the model. To find such a path one must first shut down the shocks by setting $u_t^i = 0$ for $i = 1, 2$. The fact that the productivity shock is stationary then implies that $V_t = 1$. Even when there are no shocks to the model, there may still be no balanced growth path if policy is nonstationary; for some policy rules a steady state in transformed variables will exist, for others it will not. In the following analysis I will analyze the existence of a steady state when the policy rule takes the form of a fixed interest rate, the rule described in equation (30). Given such a policy one may define the nonstochastic balanced growth path of the model, $\{c^*, y^*, m^*, L^*, \mu^*, k^*\}$ to be a solution to the equations:

$$y^* = (L^*)^{\alpha}(k^*)^{1-\alpha} \,, \tag{48}$$

$$c^* \frac{U_L\left(1, \dfrac{\mu^* m^*}{c^*}, L^*\right)}{U_c\left(1, \dfrac{\mu^* m^*}{c^*}, L^*\right)} = \alpha \frac{y^*}{L^*} \,, \tag{49}$$

$$\frac{U_m\left(1, \dfrac{\mu^* m^*}{c^*}, L^*\right)}{U_c\left(1, \dfrac{\mu^* m^*}{c^*}, L^*\right)} = \bar{I} \,, \tag{50}$$

$$k^*(\gamma - 1 + d) - y^* + c^* = 0 \,, \tag{51}$$

$$(1 - \bar{I}) = \frac{\beta}{\gamma^{-\rho}} \frac{1}{\mu^*} , \tag{52}$$

$$1 = \frac{\beta}{\gamma^{-\rho}} \left[1 - d + (1 - \alpha) \frac{y^*}{k^*} \right] . \tag{53}$$

In appendix A I show that when the utility function is given by the parametric class described by equations (19)–(22) there exists a unique solution to equations (48)–(53). One may also show that there is an open set of the parameter space for which this solution is indeterminate in the sense that close to the balanced growth path there exists a continuum of rational expectations equilibria.

10. VECTOR NOTATION

In this section I will introduce a vector notation that will enable me to write the equations of the model in a more compact form. In the body of the paper I will work with these vector equations and in appendix A I define the elements of each of the various coefficient matrices.[10] My goal is to demonstrate that business cycles in this model can be described as solutions to a linear stochastic model in which the variables are deviations of each of the state variables from their balanced growth path. The stochastic elements that drive these equations will be of two kinds. First there are *fundamental* disturbances. These are the variables $\{u^1, u^2\}$ that represent the innovation to the productivity shock and the policy disturbance. Secondly, in models in which there is an indeterminacy in the equations that describe the equilibrium of the nonstochastic model, there may be a role for *nonfundamental* disturbances, also known as sunspots or animal spirits. I will introduce two sunspot disturbances that I will refer to as e_t^1 and e_t^2.

To reduce the complexity of the notation I will employ the following definitions of vectors of variables, each expressed in the form of proportional deviations from the balanced growth path. These variables, in deviation form, are defined below:

$$\tilde{c}_t = \frac{(c_t - c^*)}{c^*}, \tilde{k}_t = \frac{(k_t - k^*)}{k^*}, \tilde{I}_t = \frac{(I_t - \bar{I})}{\bar{I}}, \tilde{v}_t = \frac{(V_t - 1)}{1}, \tilde{\mu}_t = \frac{(\mu_t - \mu^*)}{\mu^*} , \tag{54}$$

$$\tilde{y}_t = \frac{(y_t - y^*)}{y^*}, \tilde{L}_t = \frac{(L_t - L^*)}{L^*}, \tilde{m}_t = \frac{(m_t - m^*)}{m^*} . \tag{55}$$

The five variables, consumption, capital, the interest rate, the productivity shock, and the money growth factor, are the *state variables* and I will refer to the vector of state variables with the notation upper case Z. The three variables, GDP, employment, and real balances (the ratio of *lagged* money to price), are *subsidiary variables* that can be written as functions of the state and I will refer to the vector of

10. The Gauss code for computing equilibria of the model analytically is available from the author's internet site at http://econweb.sscnet.ucla.edu/rfarmer as an appendix to the working paper version of this study.

these three subsidiary variables with the notation upper case X. The definitions of Z and X are given in equations (56) and (57). The definition of the vector u in equation (58) collects the fundamental disturbances together into a single term.

$$Z_t = \begin{bmatrix} \tilde{c}_t \\ \tilde{k}_t \\ \tilde{I}_t \\ \tilde{V}_t \\ \tilde{\mu}_t \end{bmatrix} \quad (56) \qquad X_t = \begin{bmatrix} \tilde{y}_t \\ \tilde{L}_t \\ \tilde{m}_t \end{bmatrix} \quad (57) \qquad u_t = \begin{bmatrix} u_t^1 \\ u_t^2 \end{bmatrix} \quad (58)$$

In addition to the fundamental disturbances $\{u^1, u^2\}$ it will also be useful to have a notation to refer to the conditional forecast errors of each of the variables of the model and to two linear combinations of these forecast errors that represent the sunspot variables e^1 and e^2. The conditional forecast errors are defined in equation (59). I have used the index variable x in this definition to represent an element of the vector $\{c, k, I, V, \mu, y, L, m\}$.

$$\varepsilon_t^x = [x_t - E_{t-1}\{x_t\}] . \tag{59}$$

Using this definition one can collect together the vectors of conditional forecast errors associated with each of the vectors variables, Z and X;

$$\varepsilon_t^z = \begin{bmatrix} \varepsilon_t^c \\ \varepsilon_t^k \\ \varepsilon_t^I \\ \varepsilon_t^V \\ \varepsilon_t^\mu \end{bmatrix}, \quad (60) \qquad \varepsilon_t^x = \begin{bmatrix} \varepsilon_t^y \\ \varepsilon_t^L \\ \varepsilon_t^m \end{bmatrix}. \quad (61)$$

I will show in the following analysis that the equilibria of the model can be represented as a vector difference equation in the five variables c, k, I, V, and μ. But although the dynamical system that describes the equilibria of this system has dimension five, only three of the variables, capital, the productivity shock, and the rate of interest, are predetermined. Consumption and money growth are free to move each period in a way that depends on the forward-looking expectations of the families in the model. When the steady state of the system is locally stable—the case that I will study in this paper—there are two dimensions of indeterminacy. These two dimensions of indeterminacy imply that agents are free to form beliefs in which consumption and money growth are adjusted each period in line with extraneous variables that I will refer to as "sunspots." By modeling the process for the sunspot shocks, and by restricting agents to form beliefs in the same way each period, one can resolve the indeterminacy and generate an economic model in which the covariance properties of the data are uniquely determined. To represent the two sunspot shocks in the model I will use the notation e^1 and e^2. Equation (62) collects these two shocks into a single vector, e;

$$e_t = \begin{bmatrix} e_t^1 \\ e_t^2 \end{bmatrix} . \tag{62}$$

11. LINEARIZING THE MODEL

In this section I will write down a linear form of the equations of the model by taking a Taylor series expansion around the balanced growth path. Using the notation that I developed in the previous section one can write the equations of the model, (40)–(47), as approximate linear equations:

Production function,

$$\tilde{y}_t - \alpha\tilde{L}_t - (1 - \alpha)\tilde{k}_t - \tilde{V}_t = 0 , \tag{63}$$

Labor market,

$$a_1\tilde{m}_t + a_1\tilde{\mu}_t + a_2\tilde{c}_t + a_3\tilde{L}_t - \tilde{y}_t = 0 , \tag{64}$$

Monetary equilibrium,

$$a_4\tilde{m}_t + a_4\tilde{\mu}_t - a_4\tilde{c}_t + a_5\tilde{L}_t - \tilde{I}_t = 0 , \tag{65}$$

Capital accumulation,

$$\tilde{k}_{t+1} + a_6\tilde{c}_t - a_7\tilde{k}k_t + a_8\tilde{y}_t = 0 , \tag{66}$$

Bond Euler equation,

$$a_9\tilde{c}_{t+1} + a_{10}\tilde{m}_{t+1} + a_{11}\mu_{t+1} + a_{12}\tilde{L}_{t+1} - a_9\tilde{c}_t - a_{10}\tilde{m}_t + a_{13}\tilde{\mu}_t - a_{12}\tilde{L}_t$$
$$- a_{10}\varepsilon^m_{t+1} - a_{11}\varepsilon^\mu_{t+1} - a_{12}\varepsilon^L_{t+1} - a_9\varepsilon^c_{t+1} + a_{14}\tilde{I}_t = 0 , \tag{67}$$

Capital Euler equation,

$$a_{15}\tilde{c}_{t+1} + a_{16}\tilde{m}_{t+1}\, a_{16}\tilde{\mu}_{t+1} + a_{17}\tilde{L}_{t+1} - a_{15}\tilde{c}_t - a_{16}\tilde{m}_t - a_{16}\tilde{\mu}_t - a_{17}\tilde{L}_t$$
$$+ a_{18}\tilde{y}_{t+1} - a_{18}\tilde{k}_{t+1} - a_{15}\varepsilon^c_{t+1} - a_{16}\varepsilon^m_{t+1} - a_{16}\varepsilon^\mu_{t+1}$$
$$- a_{17}\varepsilon^L_{t+1} + a_{18}\varepsilon^y_{t+1} + a_{18}\varepsilon^k_{t+1} = 0 \tag{68}$$

Productivity shock,

$$\tilde{V}_{t+1} - \zeta\tilde{V}_t - u^1_{t+1} = 0 , \tag{69}$$

Policy rule,

$$\tilde{I}_{t+1} + a_{19}\tilde{y}_t + a_{20}\tilde{L}_t + a_{21}\tilde{m}_t + a_{22}\tilde{c}_t + a_{23}\tilde{k}_t + a_{24}\tilde{I}_t + a_{25}\tilde{V}_t$$
$$+ a_{26}\tilde{\mu}_t - u^2_{t+1} = 0 \,. \tag{70}$$

where the coefficients a_1–a_{18} are functions of the steady-state values of the variables c^*, k^*, $\bar{I}$, μ^*, y^*, L^*, and m^* that are derived by evaluating the derivatives of the expressions in equations (40)–(47) at the balanced growth path—these parameters are defined in appendix A. The coefficients a_{19}–a_{26} define the policy rule and each of these coefficients represents the elasticity of the policy reaction function to one of the variables of the model.

12. THE MODEL IN VECTOR FORM

In vector notation one may write the equations of the model in the following way. Equations (63)–(65) form a block of three static equations that I describe in vector notation in equation (71);

$$A_1 X_t + A_2 Z_t = 0 \,. \tag{71}$$

The equations, (66)–(70), form a separate dynamic block:

$$A_3 X_{t+1} + A_4 Z_{t+1} + A_5 X_t + A_6 Z_t + A_7 u_{t+1} + A_8 \varepsilon^Z_{t+1} + A_9 \varepsilon^X_{t+1} = 0 \,. \tag{72}$$

Since A_1 is of full rank, one may use equation (71) to write the vector X as a function of Z:

$$X_t = M Z_t \,, \qquad M = -A_I^{-1} A_2 \,. \tag{73}$$

Similarly, one may obtain an expression that relates the errors e^X to the errors ε^Z.

$$\varepsilon^X_t = M \varepsilon^Z_t \,. \tag{74}$$

Using equations (73) and (74) one may replace X and ε^X in (72) and solve for a set of equations in the state variables:

$$Z_{t+1} = J_1 Z_t + J_2 u_{t+1} + J_3 \varepsilon^Z_{t+1} \tag{75}$$

where the matrices J_1, J_2, and J_3 are defined in equations (76):

$$J_1 = -(A_4 - A_3 A_1^{-1} A_2)^{-1} (A_6 - A_7 A_1^{-1} A_2) \,,$$
$$J_2 = -(A_4 - A_3 A_1^{-1} A_2)^{-1} A_7 \,,$$
$$J_3 = -(A_4 - A_3 A_1^{-1} A_2)^{-1} (A_9 - A_8 A_1^{-1} A_2) \,. \tag{76}$$

So far I have shown that the equilibria of the model must obey a set of stochastic equations in which the disturbances to these equations are of two kinds, fundamental disturbances and forecast errors. If the equilibrium of the model was unique then one would be able to solve for the forecast errors ε^Z as functions of the fundamentals by solving the unstable roots of the matrix J_1 forwards. The technique for solving rational expectations models in this way is, by now well known, and is described in some detail in the work of King, Plosser, and Rebelo (1988). In the case that I will be studying in this paper, however, all roots of J_1 are inside the unit circle and the standard approach breaks down. Instead, one is free to pick two sunspot variables each period. These sunspot variables, defined by the following equations:

$$a_9\varepsilon^c_{t+1} + a_{10}\varepsilon^m_{t+1} + a_{11}\varepsilon^\mu_{t+1} + a_{12}\varepsilon^L_{t+1} = e^1_{t+1} , \tag{77}$$

$$a_{15}\varepsilon^c_{t+1} + a_{18}\varepsilon^k_{t+1} + a_{18}\varepsilon^y_{t+1} + a_{16}\varepsilon^m_{t+1} + a_{17}\varepsilon^L_{t+1} + a_{16}\varepsilon^\mu_{t+1} = e^2_{t+1} , \tag{78}$$

are equal to the forecast errors of the two expectational Euler equations, (44) and (45). In addition to the equations (77) and (78) one also has information about three of the elements of the vector ε^Z that must hold in a rational expectations equilibrium,

$$\varepsilon^k_{t+1} = 0 \quad (79) , \qquad \varepsilon^V_{t+1} = u^1_{t+1} \quad (80) , \qquad \varepsilon^I_{t+1} = u^2_{t+1} . \tag{81}$$

Equation (79) says that the forecast error on capital is zero. This follows from the fact that capital at date $t + 1$ is known at date t. Equations (80) and (81) state that the forecast errors for the productivity shock V and for the interest rate I are equal to the true disturbances; these equations are an implication of the rational expectations assumption which implies that agents know the probability distributions of the fundamental shocks. In matrix form, equations (77)–(81) can be written as a single equation that describes the conditional forecast errors ε^Z as a function of the fundamental errors u and of the two sunspot errors, e:

$$\epsilon^Z_{t+1} = H_5 u_{t+1} + H_6 e_{t+1} , \tag{82}$$

where the matrices H_5 and H_6 are defined in appendix A. Using equation (82) one can rewrite the stochastic difference equation (75) as follows:

$$Z_{t+1} = J_1 Z_t + J_4 u_{t+1} + J_5 e_{t+1} , \tag{83}$$

where the details of the algebra, together with the definitions of the matrices J_4 and J_5 are again explained in appendix A. In the following section I will discuss the role of indeterminacy in this model and I will show how one might compute a solution.

13. INDETERMINACY AND EQUILIBRIUM

In this discussion I will focus on the case of a perfect foresight model in which the policy rule is to set the rate of interest equal to a constant. In this case the dynamic equations (83) can be reduced to a nonstochastic system of difference equations in the three state variables, c, k, and μ. Let us write this system as

$$\begin{bmatrix} \tilde{c}_{t+1} \\ \tilde{k}_{t+1} \\ \tilde{\mu}_{t+1} \end{bmatrix} = Q \begin{bmatrix} \tilde{c}_{t} \\ \tilde{k}_{t} \\ \tilde{\mu}_{t} \end{bmatrix}, \tag{84}$$

where Q is the subset of J_1 associated with the variables c, k, and μ. It is relatively easy to show that, under interest rate control, the matrix Q has the special structure:

$$Q = \begin{bmatrix} Q_1 & 0 \\ q_2 & 0 \end{bmatrix} \tag{85}$$

where Q_1 is a 2×2 matrix associated with the real variables c and k and q_2 is a 1×2 vector that feeds the changes in the real economy back to the money growth rate. Since the matrix Q has a zero root, interest rate control leads to indeterminacy of the price level. The real variables c and k are determined by the upper left block of the system (85), but the initial value of money growth rate is free. Since the value of real balances (the ratio of lagged money to price) can be described as a function of c and k, the indeterminacy of the money supply implies that the price level also is indeterminate.

The fact that interest rate control may lead to indeterminacy is relatively well understood and has been widely discussed by previous authors, although typically the literature on the indeterminacy of equilibrium when the government follows a fixed interest rate rule has focussed on the case in which the real part of the system is determinate; in other words, most existing literature has studied the case in which Q_1 has two roots that split around unity in absolute value. The fact that these roots split around unity enables one to use standard techniques to solve the unstable root of the system forward to find consumption as a function of capital. In this case, although the price level is indeterminate, interest rate control leaves the real variables of the economy uniquely determined. The novelty in this paper is to parameterize the economy in such a way that the roots of Q_1 are both inside the unit circle and thus there is a *real* indeterminacy in addition to the nominal indeterminacy that follows from interest rate control.[11]

11. The reason for this real indeterminacy is somewhat different from the cases that have been studied in recent work by Benhabib and Farmer (1994), Gali (1994), Beaudry and Devereux (1993), and other authors who rely on increasing returns to scale, either through decreasing marginal costs as in Benhabib and Farmer (1994) and Beaudry and Devereux (1993) or through markups and fixed costs that maintain zero profits as in Gali (1994). In all three of these papers the indeterminacy is a feature of the real economy. In contrast, in the model that we are analyzing in this paper we assume that the technology satisfies constant returns to scale and the indeterminacy follows purely from an interaction between the real and nominal parts of the economy. In our model, if there were no money, the model would collapse to a standard real business cycle economy.

Some intuition for indeterminacy and a sketch of a proof can be gleaned by studying the static equations (63)–(65). In the following discussion I will focus on the special case of the utility function described by equation (86).[12]

$$U = \frac{C^{1-\rho}}{1-\rho} - \left(\frac{M}{P}\right)^{1-\rho} L^{1+\chi} . \tag{86}$$

If one uses the production function, equation (63), to eliminate GDP from equations (64) and (65), one can think of the resulting two equations as demand and supply equations for labor and for money. Consider first, equation (64) (the first-order condition for labor) and suppose that the labor market is decentralized. In this case one can think of a household equating the slope of its indifference curve to the real wage and a firm equating the real wage to the marginal product of labor. The log linear form of these demand and supply equations, for the utility function given by (86), are given in equations (87) and (88):

$$(1 - \rho)\log(\tilde{m}) + \rho\log(c) + \chi\log(L) = \log(\omega) \tag{87}$$

$$(1 - \alpha)\log(k) + (\alpha - 1)\log(L) = \log(\omega) . \tag{88}$$

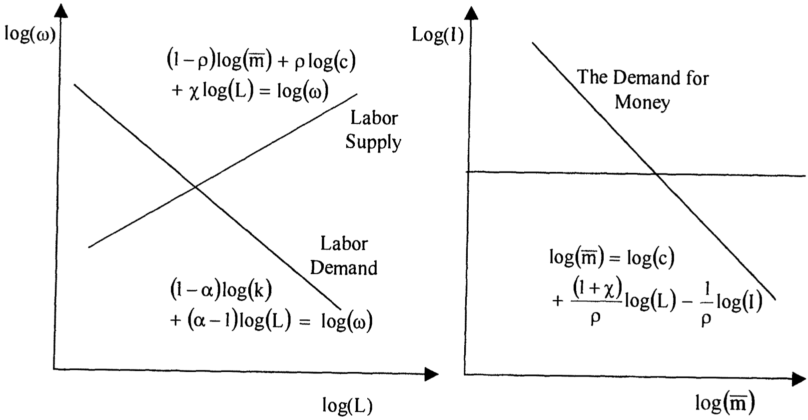
In equations (87) and (88) I have used the symbol $\tilde{m}$ to mean real balances $(m + \mu)$ and I have dropped the constant terms. Equation (87) can be thought of as a labor supply curve that is shifted by consumption and by real balances. Equation (88) can be thought of as a labor demand curve that is shifted by the stock of capital. These two curves are plotted in Figure 1A. Figure 1B graphs the money market equilibrium equation,

$$\log(\tilde{m}) = \log(c) + \frac{(1 + \chi)}{\rho}\log(L) - \frac{1}{\rho}\log(I) , \tag{89}$$

for our special case of the utility function. If we combine the labor market with the monetary equilibrium condition, it is possible to eliminate real balances from the labor market entirely. The resulting equation,

$$\log(c) - \frac{(1 - \rho)}{\rho}\log(I) + \left(\frac{1 + \chi}{\rho} - 1\right)\log(L) = \log(\omega) , \tag{90}$$

12. The utility function described in equation (86) is rich enough to display indeterminate equilibria and to illustrate the mechanism by which it occurs. It is not rich enough to fit all of the features of the data, however, since, when the function in (86) is parameterized in a way that displays an indeterminate equilibrium, it implies that the share of social resources lost by the use of money should be of the order of 50 percent of GDP. The actual number is closer to 1 percent. Equation (86) also implies that the interest elasticity of the demand for money should be equal to one whereas estimates by Hoffman, Rasche, and Tieslau (1995), based on the cointegrating coefficient of the rate of interest and the velocity of circulation, suggest that this parameter should be closer to one-half. The complications introduced in equations (19)–(22) retain the possibility of indeterminacy but allow the model to capture these additional features of the data.



FIGS. 1A (left) and 1B (right). The Structural Demand and Supply Functions for Labor and Money

is a reduced-form labor supply equation that combines money demand with the structural labor supply equation. Equation (90) is graphed in Figure 2 along with labor demand, equation (88).

Benhabib and Farmer (1994) studied a real model that is very similar to the model in this paper and they showed that a necessary and sufficient condition for the roots of the corresponding dynamical system (the analog of Q_1) to be inside the unit circle was that the slope of the labor demand curve should be greater than the slope of the labor supply curve. They demonstrated that one way in which this condition could hold would be if the labor demand curve were to slope up because of externalities in production. In the present context this would be equivalent to assuming that the parameter α is greater than one.

The magnitude of the increasing returns required in the Benhabib-Farmer paper has been criticized by a number of authors since returns to scale just don't seem to be that big.[13] However, a similar route to indeterminacy follows in a monetary model in which the production function satisfies constant returns to scale. In the configuration of reduced-form labor demand and supply curves in Figure 2, the reduced-form labor supply curve slopes down because the structural labor supply curve is shifted by real balances. The two reduced-form equations (89) and (90) have exactly the same structure as the Benhabib-Farmer real model and when the utility function is parameterized as in equation (86), the dynamic equations of the model also have the same structure. It follows that the condition for indeterminacy remains the same; that is, the labor demand and supply curves should cross with the "wrong slopes" and the labor supply curve should be steeper than labor demand. Whereas in the real model this occurs when the labor demand curve slopes up, be-

13. See for example the papers by Basu and Fernald (1995) and Burnside, Eichenbaum, and Rebelo (1995).

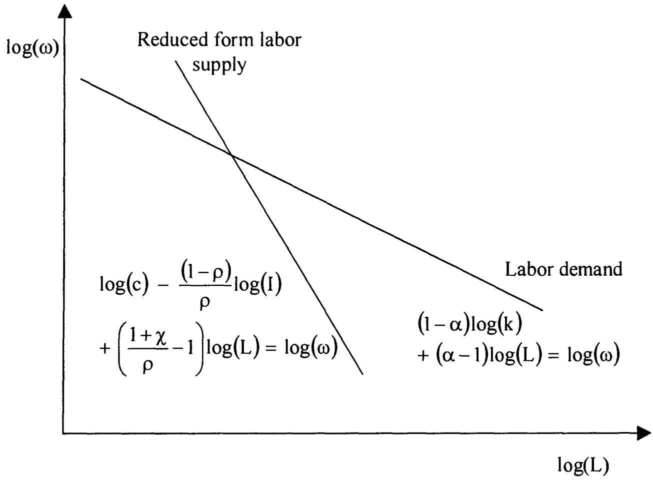


FIG. 2. The Reduced-Form Demand and Supply Functions for Labor

cause of externalities in the monetary model, demand and supply can cross with the wrong slopes because the reduced-form labor supply curve *slopes down.*[14] The parameter configuration that leads to this possibility is given by inequality (91):

$$1 - \alpha < \left(1 - \frac{1 + \chi}{\rho}\right). \tag{91}$$

When α is equal to 2/3, indeterminacy requires a small value of χ (elastic labor supply) and a value of ρ well within the range that is considered acceptable from estimates of the intertemporal elasticity of substitution reported in the literature on calibration. For example, if one sets $\chi = 0$, the value consistent with Gary Hansen and Richard Rogerson's model of indivisible labor, indeterminacy occurs if the parameter ρ is greater than 1.5.

14. CALIBRATING THE MODEL

I have argued that a model close in form to the RBC model can display indeterminate equilibria and I have suggested that a model, parameterized in this way, might help one to explain the data. In this section I am going to calibrate a model of this form and I am going to use the calibrated model to simulate data and to compare the simulated data with actual data from the U.S. economy. The example that I will use

14. The way that indeterminacy occurs in this example is similar to Gali (1994) in which the structural demand and supply curves also have standard slopes. In Gali's work, increased output reduces markups and shifts out labor demand so that the reduced-form labor demand curve slopes up. In contrast, in the monetary model increases in money shift out labor supply so that the reduced-form labor supply curve slopes down.

TABLE 1

Parameter Values Used in the Simulations (Real Variables)

	Value	Definition
α	0.66	Labor's share
d	0.06	Annual depreciation
β	0.928	Annual rate of time preference
γ	0.01	Per capita growth rate
ζ	0.7	Autocorrelation of productivity shock
ρ	1.42	Intertemporal elasticity of substitution
χ	0	Inverse labor supply elasticity

in the calibration exercise is more complicated than the functional form in equation (86) since one requires additional flexibility in order to capture the fact that the share of social resources lost by using money is relatively low. Table 1 lists some of the parameters that appear in real models and lists the values of these parameters that I have used in the simulations that I will report in the paper. Most of these are relatively standard, and are similar to the values used in other calibrated models. The parameters α, d, β, and γ are chosen to match long-run features of the data. The parameter χ is the inverse labor supply elasticity and the choice of $\chi = 0$ implies an infinite labor supply elasticity. It is well known that highly elastic labor supply is necessary in this class of models to match the observed employment volatility and the choice of $\chi = 0$ is typically justified by appealing to the arguments for indivisible labor in Hansen (1984) and Rogerson (1988). The choice of $\rho = 1.42$ corresponds to a model that is nonseparable in leisure and consumption—the value $\rho = 1$ would imply logarithmic preferences. A value of ρ somewhat higher than unity is necessary in this model to generate indeterminate equilibria. Table 2 lists some of the first moments of the data that are implied by the real parameterization described in table 1.

The consumption income ratio is higher than is usually reported because the consumption variable used in this study includes government consumption. The time preference factor of 0.928 was chosen to match the real interest rate of 6 percent and, as a consequence, the model reports a lower ratio of GDP to capital (0.35 as opposed to 0.44) than exists in the annual data. I have not reported labor hours in this study because the value of 1.21 reported from the model is unit dependent. The function $V(L)$ contains an arbitrary constant that can be chosen to fit any observed value of L and the choice of this constant is equivalent to picking the units of mea-

TABLE 2

Comparison of Model with Data (Parameters Chosen from Table 1)

	Model	Data	
c/y	0.81	0.80	Consumption income ratio
y/k	0.35	0.44	Income capital ratio
L	1.21		Labor supply
r	.06	.06	Real interest rate

TABLE 3

PARAMETER VALUES USED IN THE SIMULATIONS (NOMINAL VARIABLES)

	Value	Definition
a	0.0002	Share of money in X
b	0.00015	Share of money in W
λ	2.5	Substitutability of money for consumption and leisure

surement of hours; by the same reasoning, the production function can contain a normalizing constant that I have omitted in the paper.

In addition to the real parameters of the model, the utility function contains three additional parameters that determine the properties of the monetary features of the model. These parameters are reported in Table 3. To calibrate these parameters in practice I picked values for a and for λ in a manner described below and I restricted b so that the predicted values of the velocity of circulation and of the interest cost of holding money would fit the numbers reported in Table 4. I have not reported the actual data on the rate of interest and the velocity of circulation in this table because the data is nonstationary and it makes no sense to average them to obtain a single statistic.

Figure 3 graphs the data on the rate of interest (the six-month commercial paper rate) and the velocity of circulation from U.S. data from 1929 through 1988. The interest rate ranges from half of 1 percent to 13 percent and the velocity of circulation from two to seven. The model contains a stationary balanced growth path only for a fixed interest rate policy and the fact that the interest rate was increasing over the sample period implies that the balanced growth of the model was itself shifting. I have handled this nonstationarity in the data by picking an arbitrary point around which to linearize the model and I have picked this point to lie within the range of observed interest rates in the data.

A second complication arises in deciding the appropriate nominal rate of interest to use. The correct concept from the point of view of the model is the interest lost by holding M1 as opposed to some other interest-bearing asset that has no use in exchange. Since a substantial component of M1 bears interest, the six-month commercial paper rate undoubtedly overstates the opportunity cost of holding money. In calibrating the model I have erred on the side of caution by picking parameter values

TABLE 4

COMPARISON OF MODEL WITH DATA (PARAMETERS CHOSEN FROM TABLE 3)

	Model	
y/m	6	Velocity of circulation
ε_i	0.4	Interest elasticity of money
I	.01	Nominal interest rate
im/y	.0016	Resource cost of using money

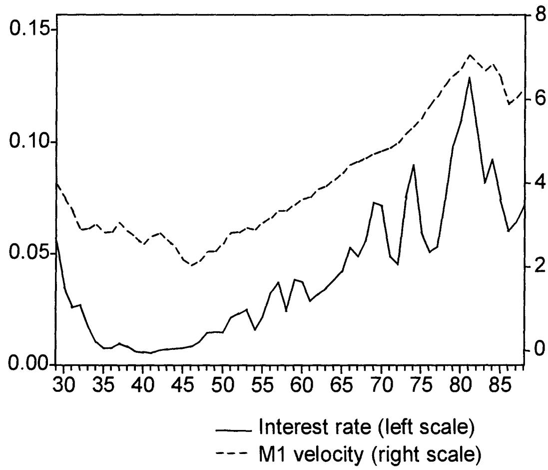


FIG. 3. The Rate of Interest and the Velocity of Circulation

that imply that the first-order effects of using money are relatively small. I chose a value for the opportunity cost of holding money of 1 percent and a value of the velocity of circulation of 6. Together, this parameterization implies that the resources lost by a representative agent through using money was equal to 0.16 percent of GDP.

Table 4 also reports a value for ε_i, the interest elasticity of the demand for money. To compute this statistic for the model one can show that, in the steady state, the velocity of circulation predicted by the model is a function of the rate of interest. The number ε_i reported in Table 4 is the elasticity of this function evaluated along the balanced growth path. The corresponding statistic reported for the data comes from recent evidence of the demand for money that is based on the long-run properties of velocity and the rate of interest. Recent work, for example, by Hoffman, Rasche, and Tieslau (1995) places the value of ε_i at around 0.5.

The process by which I calibrated the model parameters was to pick values for λ and a and to fix b so that money's share of GDP (iy/m) would be equal to 0.16 percent along the balanced growth path. I then used a computer to simulate impulse response functions for four variables in a manner described in the following section. Given the theoretical impulse response functions, I experimented with different values of χ, ρ, a and λ to match the observed impulse response functions as closely as possible whilst keeping the interest elasticity of the velocity of circulation (the statistic ε_i) within the range 0.3–0.6. In theory ε_i depends on both a and λ, but in the relevant part of the parameter space it is determined mainly by the parameter λ that measures the elasticity of substitution between money and consumption and labor supply.

To fix the parameters of the interest rate rule I experimented with two different sets of assumptions. The first was to set all of the feedback coefficients, a_{19}–a_{26}

equal to zero. The second was to estimate a regression of the rate of interest on lagged values of the endogenous variables in time series data and to use these estimated coefficients in the model. In practice, the only coefficient that made much of a difference to the dynamic properties of the simulated data was the coefficient a_{24} that represents the response of the interest rate to its own past values. In the simulations reported below I set a_{24} equal to its estimated value of -0.5 and I set all of the other coefficients of the policy rule to zero.

In practice the shape of the reported impulse response functions was not overly sensitive to the choice of λ and I chose a value of $\lambda = 2.5$ to fix ε_i at 0.4. Given this choice of λ I found that the shape of the impulse response functions in the data was extremely sensitive to the choice of ρ and a. The parameter a is restricted to a relatively small range from around 0 to 0.0004 in order to maintain a positive value of b and to simultaneously match the money share statistic (im/y) of 0.16 percent. The following section describes the way that I computed impulse response functions from the model and it describes data that was generated for the parameter choice reported in Tables 1 and 3.

15. COMPUTING VECTOR AUTOREGRESSIONS

This section describes how I picked the variance covariance matrix of the shocks to the model. In the subsequent section I discuss the methods that I used to pick the key parameters ρ and a. The choice of these parameters was geared to address a particular question. Suppose that an econometrician were to be given a particular subset of the data generated by my model; how closely would the artificial data resemble the actual time series on the same subset of variables? The subset that I wish to study is one that was originally investigated by Chris Sims: the U.S. data on price, GDP, the rate of interest, and the stock of money. The analogues of these variables that I will use in this discussion are the variables, s_t, defined below:

$$s_t = \begin{bmatrix} \tilde{y}_t \\ \tilde{m}_t \\ \tilde{I}_t \\ \tilde{\mu} \end{bmatrix} . \tag{92}$$

These variables can be described as a linear function of the state vector:

$$s_t = D_1 Z_t + D_2 X_t = (D_1 + D_2 M) Z_t = D_3 Z_t \tag{93}$$

where the matrices D_1, D_2, and D_3 are defined in appendix A. The vector s_t consists of deviations from the balanced growth path of GDP, real balances (lagged money divided by current price), the rate of interest, and the money growth rate.

Let us suppose that an econometrician were to run a vector autoregression using two lags of GDP, real balances (defined as in the model), the rate of interest, the rate

TABLE 5

ESTIMATED VARIANCE COVARIANCE MATRIX OF RESIDUALS × 10^4

Estimated VCV Matrix of Residuals × 10^4	GDP	Real Balances	Interest Rate	Money Growth
GDP	19.17	2.00	0.618	0.798
Real Balances	2.00	10.00	−0.33	−3.26
Interest Rate	0.618	−0.33	0.787	−0.602
Money Growth	0.798	−3.26	−0.602	14.39

of money growth, and a time trend.[15] Let the estimated variance covariance matrix of the residuals from an autoregression of this form be denoted Ω_s. The estimated values of Ω_s from a VAR of this form on postwar U.S. data is given in Table 5.

If the data were generated by the model described in this paper, then one could recover the VCV matrix of the underlying shocks from the VCV matrix of the residuals, since the true errors on the equations of the vector autoregression are a linear function of the error vector $\{u, e\}$. The relationship between the VCV matrix of $\{u, e\}$ and the VCV matrix of the true residuals in a VAR is given by equation (94):

$$\Omega = (D_4)\Omega_s(D_4^T)^{-1}\ , \tag{94}$$

where the superscript T denotes transposition and the definition of the matrix D_4 is given in the appendix.

Table 6 reports the VCV matrix of the shocks that is implied by my estimate of a vector autoregression for the parameterization reported in Tables 1 and 3. In both cases Tables 5 and 6 multiply the actual numbers by 10^4. This table can be used to interpret the residuals to the VAR in terms of the corresponding structural shocks. Notice that three of the diagonal elements of this matrix are relatively large and of a comparable order of magnitude; these diagonal elements represent the productivity shock and the two sunspot shocks. The fact that these three elements are large implies that the model ascribes a relatively important role to the productivity shock, but it also implies an important role for sunspots. In addition to relatively large diagonal elements for each of the sunspot shocks notice that these two errors are also highly correlated with the fundamental shock u^1. The fact that the off-diagonal elements that represent these shocks are also large implies that individuals "overreact" to fundamentals.

16. SIMULATED DATA

This section reports some sample statistics from annual U.S. data and it compares these moments with sample statistics from a single series of sixty observations sim-

15. Since the set s_t does not contain observations on the autocorrelated productivity shock, a first-order vector autoregression would not be sufficient to recover the dynamics of the underlying model. However, by rewriting the unobservable variable V_t as a function of the first difference of the observable variables L_t, Y_t, and K_t, one can show that a vector autoregression with two or more lags *would* recover the underlying dynamics.

TABLE 6

VARIANCE COVARIANCE MATRIX OF THE ERRORS × 10^4

VCV matrix of Errors × 10^4	u^1 Productivity Shock	u^2 Policy Shock	e^1 First Sunspot	e^2 Second Sunspot
u^1	17.11	−0.604	17.70	24.50
u^2	−0.604	0.787	−0.522	−0.837
e^1	17.70	−0.522	25.81	25.40
e^2	24.50	−0.837	25.40	35.09

ulated from the model. The simulated data was driven by a vector of four normal i.i.d. random variables with the variance covariance matrix Ω, described in Table 6. Once the variance covariance matrix was chosen, I was still faced with the task of choosing the parameters a, b, ρ, and λ. There are three moments that are governed by these parameters. First, ρ governs the correlation that one would expect to see between consumption growth and asset market returns. Attempts to estimate ρ from this source have led to wildly conflicting evidence in the literature depending on whether one uses bond market returns or equity. There is some consensus that ρ is greater than unity but little evidence on exactly how large it should be.

The parameters a, b, and λ together govern the share of interest payments in GDP and the interest elasticity of the demand for money. I have already argued that the interest cost of holding money is of the order of 1 percent and that the interest elasticity of the demand for money is approximately one half. However, there is little consensus as to the exact magnitude of either of these moments. Since there is room for reasonable people to differ, my procedure was to choose the parameters ρ, a, b, and λ to give myself as much flexibility as possible to match the impulse response functions in actual data. In practice the exact shape of the impulse response functions in simulated data is sensitive to all of these parameters. Indeterminacy occurs for a very wide range of values. In section 18 I will report the results of a sensitivity analysis that illustrates just how wide the bands are within which the model displays a fully stable steady state.

The benchmark parameters that I chose were values of $\rho = 1.42$, $\lambda = 2.5$, $a = .0002$, and $b = .00015$. These choices gave the best fit to the actual impulse responses, where the goodness-of-fit criteria that I used were based on my judgment that the simulated impulse responses were within the standard error bounds of the data in as many cases as possible. This procedure could clearly be automated; it amounts to a "back of the envelope" simulated method of moments estimator. Since, in developing this model there was a considerable degree of experimentation it did not seem worthwhile using more formal estimation techniques at this point.

The parameters a and b were not chosen independently since I wanted to restrict the share of resources lost in using money to be small. In the simulations I chose the opportunity cost of holding money to equal 1 percent and the velocity of circulation to equal 6 which implies that the resource cost of holding money is 0.16 percent. This number was used to fix the value of b for any given value of a. In practice a had to be chosen very small to come close to this resource cost.

TABLE 7

COMPARISON OF FIRST MOMENTS OF SIMULATED AND ACTUAL DATA

	Simulation		Data	
	standard deviation	s.d. relative to GDP	standard deviation	s.d. relative to GDP
Consumption	0.09	0.47	0.09	0.75
Capital	0.22	1.15	0.08	0.66
Interest rate	0.01	0.05	0.02	0.16
Productivity shock	0.04	0.21	0.07	0.58
Money growth	0.04	0.21	0.06	0.5
GDP	0.19	1	0.12	1
Employment	0.20	1.05	0.08	0.66
Real balances	0.10	0.52	0.25	2.08

Table 7 reports the standard deviations of the variables used in the study and their standard deviations relative to GDP together with the standard deviations of a single simulation of the model. Figures 4 through 10 graph the U.S. data and the model simulation. These graphs give an impression of the dimensions in which the model succeeds at generating business cycles, and the dimensions at which it fails. I have assumed implicitly that the data is generated by the same process for the entire period from 1929 through 1988. In fact, there is considerable evidence that this assumption is false. The magnitude of the residuals in the prewar period is quite a bit greater than in the postwar period.

The simulated data behaves a lot more like the prewar business cycles, at least in magnitude. Consumption, in the simulations, is too smooth and it is not as highly correlated with GDP as the actual data. Employment, on the other hand, is too volatile. Capital has the same long slow swings in the simulation that we see in the data, but it has the wrong phase and is much too volatile. Figure 5 illustrates that in the U.S. data, capital lags GDP whereas in the long swings in the simulated data it leads the cycle.

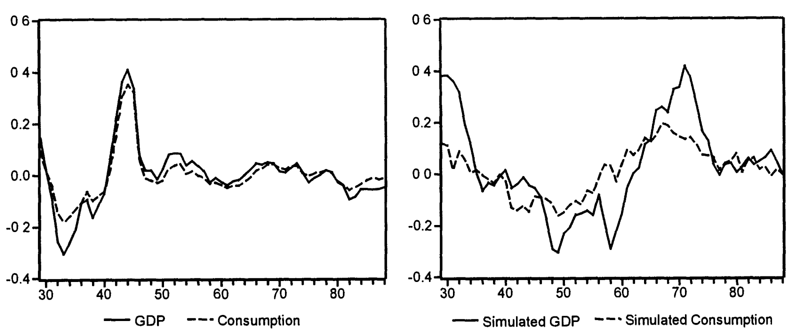


FIG. 4. Simulated Actual Consumption and GDP

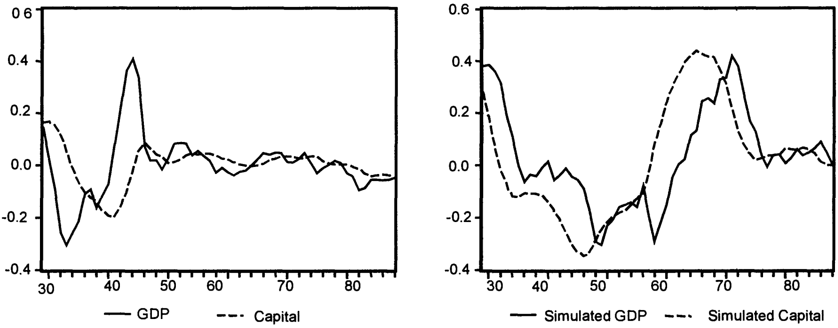


FIG. 5. Simulated and Actual Capital and GDP

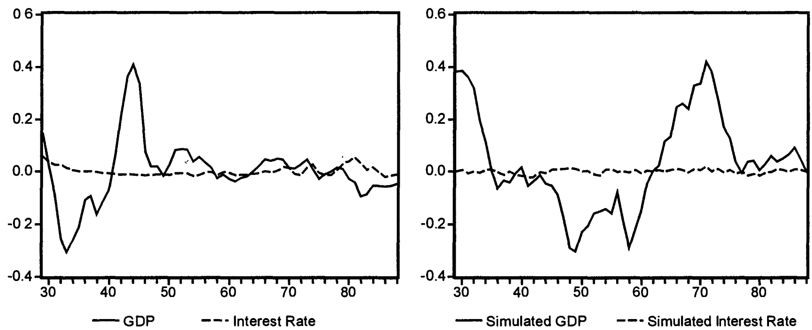


FIG. 6. Simulated and Actual Interest Rate and GDP

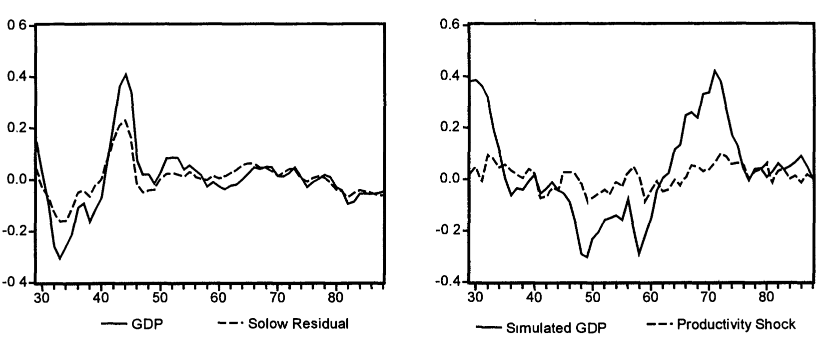


FIG. 7. Simulated Productivity Shock, the Solow Residual, and GDP

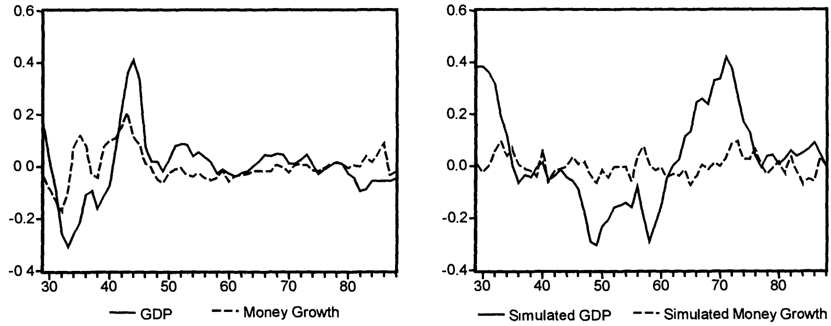


FIG. 8. Simulated and Actual Money Growth and GDP

FIG. 9. Simulated and Actual Employment and GDP

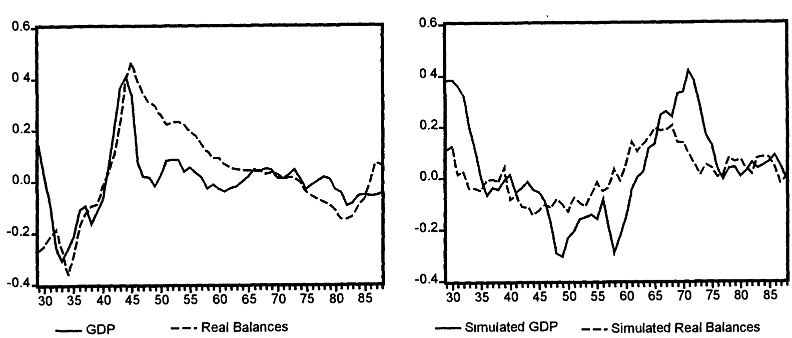


FIG. 10. Simulated and Actual Real Balances and GDP

17. IMPULSE RESPONSE FUNCTIONS

To get a better understanding of how the dynamic properties of the simulation compare with the dynamic properties of the data, I have computed the theoretical impulse response functions that would be observed in a four-variable vector autoregression using data on GDP, real balances, the interest rate, and money growth. I have picked one particular ordering, $\{Y, m, I, \mu\}$, and I have plotted the theoretical impulse response functions computed from the model on the same figure as the estimated impulse response functions from U.S. data. The VAR for the United States used two lags of each of the variables together with a time trend and the estimated residuals from the VAR were used to compute the theoretical variance covariance matrix of the model shocks as reported in Table 6.

Figures 11–14 report the results of this experiment. In each case the solid line is the estimated impulse response function and the short dashed lines represent two standard error bounds. The lines with long dashes are the theoretical impulse response functions from the model. Notice that, in every case, the theoretical impulse response functions and the estimated impulse response functions begin at the same point; this is by construction since I have chosen the variance covariance matrix of the shocks in such a way that the model generates residuals for the simulated data that has exactly the estimated variance covariance matrix as the data. Seven of the sixteen model impulse response functions lie within two standard error bounds from the data in every year, a further four lie within these bounds 70 or 80 percent of the time. Even the impulse response functions that do relatively poorly have the right qualitative shape. Consider, for example, the response of GDP to a GDP shock (the top left panel of Figure 11). In this graph the simulated impulse response function lies outside of the two standard error bounds in every year. Nevertheless, the broad

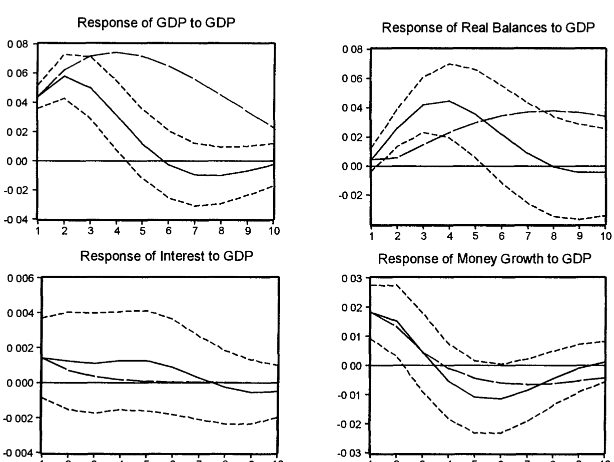


FIG. 11. Impulse Response Functions to a GDP Shock

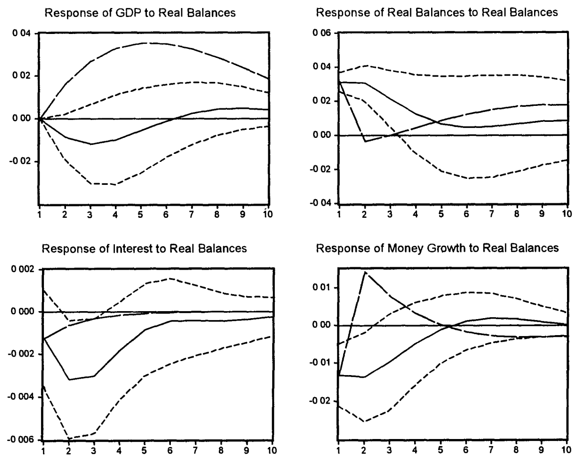


FIG. 12. Impulse Response Functions to a Real Balance Shock

qualitative picture is correct; GDP increases slowly to peak and then cycles back to the balanced growth path. In my experiments it proved possible to choose parameters in a way that would match the impulse response to a GDP shock perfectly by lowering the parameters a and ρ. The cost of this, however, was to make the fit of the impulse response functions to money growth and to real balances much worse. The numbers that I reported in the final simulation were a compromise that was designed to minimize an implicit metric that paid attention to all four sets of graphs.

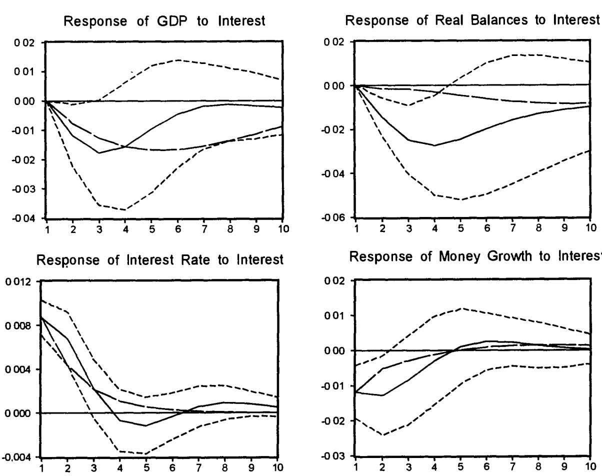


FIG. 13. Impulse Response Functions to an Interest Rate Shock

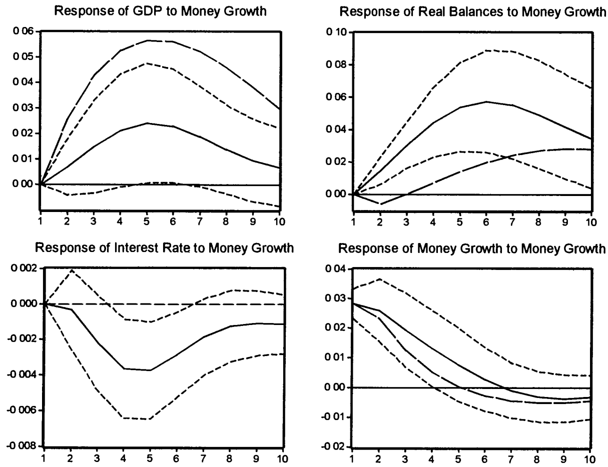


FIG. 14. Impulse Response Functions to a Money Growth Shock

18. SENSITIVITY ANALYSIS

In this section I report the results of a sensitivity analysis in which I vary the parameters χ, λ, and ρ and tabulate the values of the eigenvalues of the matrix J_1. Table 8 reports the results of varying the inverse labor supply elasticity, the parameter χ, from zero up to 1.25; all other parameters are set at the values described in Tables 1 and 3; we refer to these values as the benchmark case.

The table reports the roots of J_1. The case studied is one of interest rate control in which the interest rate is equal to a constant; consequently two of the roots of J_1 are identically zero. Columns 2 through 6 report the values of the roots. Column 7 is the elasticity of the demand for money function predicted by the model and column 8 reports the value of the parameter b that must be chosen to maintain the opportunity cost of holding money equal to 0.16 percent of GDP. Notice from Table 8 that the

TABLE 8

SENSITIVITY OF THE ROOTS TO CHANGING χ

χ	Roots 1 and 2	Modulus	Root 3	Root 4	Root 5	ε_1	b
0	0.88 ± .19i	0.89	0.7	0	0	0.4	0.0005
0.25	0.96 ± 0.12i	0.97	0.7	0	0	0.4	0.0006
0.5	0.97 ± .09i	0.98	0.7	0	0	0.4	0.00065
0.75	.097 ± 0.1i	0.98	0.7	0	0	0.4	0.0007
1	0.88 ± 0.19i	0.9	0.7	0	0	0.4	0.0008
1.04	0.75 ± .02	0.78	0.7	0	0	0.4	0.0008
1.25	1.29	1.29	0.7	0	0	0.4	0.0009
	0.87	0.87					

NOTE: All other parameters are as described in Tables 1 and 3.

TABLE 9

SENSITIVITY OF THE ROOTS TO CHANGING λ

χ	Roots 1 and 2	Modulus	Root 3	Root 4	Root 5	ε_i	b
2.75	0.75 ± 0.2i	0.78	0.7	0	0	0.35	0.0002
2.5	0.75 ± 0.2i	0.78	0.7	0	0	0.4	0.0008
2.25	0.75 ± 0.2i	0.78	0.7	0	0	0.44	0.0016
2.0	0.75 ± 0.2i	0.78	0.7	0	0	0.5	0.0026
1.75	0.75 ± 0.2i	0.78	0.7	0	0	0.6	0.004
1.5	0.75 ± 0.2	0.78	0.7	0	0	0.67	0.006

NOTE: χ is set at 1.04 in Table 9. All other parameters are as in the benchmark case.

matrix J_1 has two zero roots; one of these reflects the assumption that the policy rule has no autoregressive component (for the purpose of generating the table the policy rule fixes the interest rate). The second zero root is a consequence of the assumption of interest rate control; an assumption that reduces the order of the dynamics in the way discussed in section 13. The three nonzero roots of J_1 are inside the unit circle for values of the inverse labor supply elasticity from zero through 1.04. One of these roots is real and equal to 0.7; this reflects the assumption that the productivity shock is autocorrelated. The other two nonzero roots are complex and inside the unit circle for values of χ between zero and 1.04. After 1.04 there is a bifurcation and a pair of complex roots changes into a pair of real roots, one with modulus greater than one and one with modulus less than one. Indeterminacy occurs when all three roots have modulus less than one. The important point is that indeterminacy holds even when the labor supply elasticity is equal to one and it does not rely on the assumption of infinitely elastic labor.

In Table 9 I maintain the value of the labor supply elasticity at 1.04 and vary the parameter λ between 2.75 and 1.5. Notice that the roots are not sensitive to variations in λ; however, the interest elasticity of the demand for money *is* influenced by λ; it varies from $\varepsilon_i = 0.35$ when $\lambda = 2.75$, to $\varepsilon_i = 0.67$ when $\lambda = 1.5$. The parameter b also is sensitive to changes in λ.

In Table 10 I allow ρ, the intertemporal elasticity of substitution parameter, to vary between 1.3 and 2.3 holding all other parameters fixed at their benchmark values. For very low values of ρ there is a pair of positive real roots both less than one. As ρ is increased a pair of complex roots appears with modulus less than one. Increasing ρ further increases the modulus of this pair of roots from 0.88 when ρ is equal to 1.4 to 0.98 when ρ equals 2. As ρ increases beyond two the roots become real again. Initially both roots are less than unity but as ρ is increased beyond 2.3 one roots switches sign. Once again there is indeterminacy in this model for a wide range of values of ρ. In the chosen parameterization ρ was chosen to equal 1.42 in order to keep endogenous persistence of the propagation of the shocks relatively small in line with the data. As ρ is increased, the modulus of the roots gets close to one and the model generates cycles that are much more persistent than those in actual data.

As a final check on the sensitivity of the model I looked at the roots of the matrix J_1 for the case of control of the money supply. Based on my analysis of the special

TABLE 10

SENSITIVITY OF THE ROOTS TO CHANGING ρ

ρ	Roots 1 and 2	Modulus	Root 3	Root 4	Root 5	ε_i	b
1.3	0.64 0.26	0.64 0.26	0.7	0	0	0.4	0.0006
1.4	0.86 ± 0.2	0.88	0.7	0	0	0.4	0.0005
1.5	0.92 ± 0.14	0.93	0.7	0	0	0.4	0.0004
1.6	0.94 ± 0.11	0.95	0.7	0	0	0.4	0.0004
1.7	0.96 ± 0.09	0.96	0.7	0	0	0.4	0.0004
1.8	0.97 ± 0.07	0.97	0.7	0	0	0.4	0.0004
1.9	0.97 ± 0.06	0.97	0.7	0	0	0.4	0.0003
2.0	0.98 ± .004	0.98	0.7	0	0	0.4	0.0003
2.2	0.99 0.97	0.99 0.97	0.7	0	0	0.4	0.0003
2.3	1.01 0.96	1.01 0.96	0.7	0	0	0.4	0.0003

NOTE: All other parameters are as in the benchmark case.

case discussed in section 13 I expected to find that control of the money supply causes one root to be outside the unit circle. This is indeed true when the parameters are chosen to mimic the special case of the utility function in equation (86). But it is not true in general. Under monetary control in which the money stock is fixed each period, one of the roots of the matrix J_1 is equal to zero (since there is no feedback from the economy to the policy rule) but the other four are nonzero. This is in contrast to the case of interest rate control where there is an additional zero root.

Table 11 summarizes the effects of changing the monetary policy from fixing the interest rate to fixing the value of the money stock each period. The first row of this table reflects a parameterization that represents the special utility function given in equation (86). Notice that for this case there is now only one zero root. In addition, there is a root of J_1 that is outside the unit circle. This case retains one degree of indeterminacy since there are still four roots inside the unit circle and only three predetermined initial conditions. The one degree of indeterminacy comes from the impact of money in the utility function and it is consistent with my discussion in section 13 in which I argued that a second degree of indeterminacy follows when one models monetary policy with interest rate control. What is surprising is the second row of Table 11 that lists the roots of J_1 for the benchmark parameterization of the model for a situation in which the Fed sets the money growth rate equal to a constant. In this case, the model *retains two degrees of indeterminacy,* implying that

TABLE 11

THE ROOTS OF J_1 UNDER CONTROL OF THE MONETARY BASE

	a	b	λ	Roots 1 and 2	Modulus	Root 3	Root 4	Root 5	ε'
Special Utility Function (Equation 86)	0	1	0	1.29 0.89	1.29 0.89	0.98	0.7	0	0.007
Benchmark Case	0.0002	0.0005	2.5	0.87 ± 0.19	0.89	0.97	0.7	0	0.4

the results of the paper are robust not only to a wide range of parameters that are consistent with the data but also to variations in the policy rule.[16]

19. CONCLUSION

In earlier work I have argued that general equilibrium models with indeterminate equilibria may be useful tools to help us to understand the mechanism by which real and nominal shocks are propagated over the business cycle. In this paper I have provided some quantitative support for this assertion by constructing a complete general equilibrium model and by comparing the equilibrium of the model with actual data. The model does a relatively good job of capturing the long slow humped shape responses that we see in actual data, and it accomplishes this task with a minimum of assumptions.

Why should one believe that this model is useful? I believe the most persuasive case follows from its ability to outperform the alternatives. I argued in the introduction that there are three main competing explanations of the monetary transmission mechanism: contracting models, liquidity effects models, and menu costs. All three competitors rely on a barrier of some kind to price adjustment and all three have trouble explaining the persistence of monetary shocks. In contrast, I have presented a model that differs from the RBC model in only one respect; it includes real balances as an argument of the utility function. By parameterizing utility in a way that is general enough to capture a range of monetary facts, I have shown that this standard model contains an indeterminate equilibrium. The indeterminacy of equilibrium is not a problem to be avoided by clever assumptions; it is a fact that can be exploited to explain the world.

APPENDIX A

Derivation of the Balanced Growth Path

This section contains an algorithm for computing a steady state of the model for the case when the utility function is parameterized as in equations (19)–(22). For this choice of utility function, the steady-state equations (49) and (50) take the form:

$$\frac{(1+\chi)L^{\chi}f(q^*)^{1-\rho}}{g(q^*)^{1-\rho}+L^{1+\chi}(\rho-1)f(q^*)^{\lambda-\rho}(1-b)}=\frac{\alpha y^*}{L^*c^*}, \tag{a1}$$

$$\frac{ag(q^*)^{\lambda-\rho}q^{*\lambda}+(\rho-1)L^{*1+\chi}f(q^*)^{\lambda-\rho}q^{*\lambda}b}{g(q^*)^{\lambda-\rho}(1-a)+L^{*1+\chi}(\rho-1)(1-b)f(q^*)^{\lambda-\rho}}=\bar{I}, \tag{a2}$$

16. The discussion in section 13 is factually correct, but it relies critically on the assumption of separability of the utility function. Only in the separable case can one directly apply the results of Benhabib and Farmer (1994).

where the variable q^* is defined as

$$q^* = \frac{\mu^* m^*}{c^*} \tag{a3}$$

and the functions $f(\ )$ and $g(\ )$ are defined as

$$\frac{W(c^*, \mu^* m^*)}{c^*} = f(q^*) = [(1 - b) + bq^{*\lambda - 1}]^{1/1-\lambda} , \tag{a4}$$

$$\frac{X(c^*, \mu^* m^*)}{c^*} = g(q^*) = [(1 - a) + aq^{*\lambda - 1}]^{1/1-\lambda} . \tag{a5}$$

The following steps exploit the structure of equations (a1) and (a2) to demonstrate the existence of a balanced growth path.

Step I: Use equation (52) to define μ^*:

$$\mu^* = \frac{\beta\gamma^{\rho}}{(1 - \bar{I})} . \tag{a6}$$

Step II: Define the steady-state ratio of output to capital from equation (53):

$$\frac{y^*}{k^*} = \left(\frac{\gamma^{-\rho}}{\beta} - 1 + d\right)\frac{1}{(1 - \alpha)} . \tag{a7}$$

Step III: Use equation (51) to compute the steady-state ratio of consumption to capital.

$$\frac{c^*}{k^*} = \frac{y^*}{k^*} + 1 - \gamma - d . \tag{a8}$$

Equations (a7) and (a8) also imply a value for the steady-state consumption-GDP ratio:

$$\frac{c^*}{y^*} = \left(\frac{c^*}{k^*}\right)\left(\frac{k^*}{y^*}\right) . \tag{a9}$$

Step IV: For given y^* and c^*, equations (a1) in (a2) are a pair of simultaneous equations in L^* and q^*. Rearranging (a1) it follows that

$$L^* = \left\{\frac{g(q^*)^{\lambda-\rho}\alpha(y^*/c^*)}{[(1 + \chi)f(q^*)^{1-\lambda} - (\rho - 1)(1 - b)\alpha(y^*/c^*)]f(q^*)^{\lambda-\rho}}\right\}^{1/1+\chi} . \tag{a10}$$

Substituting (a10) in (a2) and rearranging leads to

$$H(q^*) = q^{*\lambda}\frac{a}{1 - a} + q^{*\lambda}f(q^*)^{\lambda-1}\frac{\alpha(\rho - 1)(y^*/c^*)}{(1 + \chi)(1 - a)}[b - a] = \bar{I} . \tag{a11}$$

In the following analysis we treat only the case $\lambda > 1$. For this case it follows from the properties of $f(\ )$ that $H(0) = 0$, $\lim_{q \to \infty} = \infty$ and further, $H(q)$ is monotonically increasing for positive q. It follows that there exists a unique value q^* for positive $\bar{I}$. The value of L^* is then given by equation (a10).

Step V: Since $H(q^*)$ is monotonic it is invertible and one can express q^* as a function $H^{-1}(\bar{I})$. One can then solve for the velocity of circulation:

$$\frac{y^*}{\mu^* m^*} = \frac{1}{H^{-1}(\bar{I})c^*} \,. \tag{a12}$$

Note that $y^*/\mu^* m^*$, rather than y^*/m^*, is equal to velocity since m^* is the ratio of *lagged* money to the price.

Definition of the Coefficients a_1–a_{18}

The following equations define the coefficients of the linearized equations (64)–(70) in terms of the parameters of the utility function. The elements of δ and D are all evaluated at the steady state $\{c^*, y^*, m^*, L^*, \mu^*, k^*\}$.

$$a_1 = \delta_{Lm} - \delta_{cm} \qquad a_2 = 1 - (\delta_{Lm} - \delta_{cm}) \qquad a_3 = (\delta_{LL} - \delta_{cL})$$

$$a_4 = (\delta_{mm} - \delta_{cm}) \qquad a_5 = (\delta_{mL} - \delta_{cL}) \qquad a_6 = \frac{c^*}{\gamma k^*}$$

$$a_7 = -\frac{(1-\delta)}{\gamma} \qquad a_8 = -\frac{y^*}{\gamma k^*} \qquad a_9 = \rho + \delta_{cm}$$

$$a_{10} = -(1 + \delta_{cm}) \qquad a_{11} = -\delta_{cm} \qquad a_{12} = -\delta_{cL}$$

$$a_{13} = 1 + \delta_{cm} \qquad a_{14} = -\frac{\bar{I}}{1-\bar{I}} \qquad a_{15} = \rho + \delta_{cm}$$

$$a_{16} = -\delta_{cm} \qquad a_{17} = \delta_{cL} \qquad a_{18} = -\frac{\beta(1-\alpha)y^*}{\gamma^{-\rho}k^*}$$

Definitions of the Matrices A_1–A_9

This section writes out each of the terms in equations (71) and (72) in full. These definitions are to assist in constructing the Gauss code to compute an equilibrium numerically.

$$A_1 X_t = \begin{bmatrix} 1 & -\alpha & 0 \\ -1 & a_3 & a_1 \\ 0 & a_4 & a_4 \end{bmatrix} \begin{bmatrix} \tilde{y}_t \\ \tilde{L}_t \\ \tilde{m}_t \end{bmatrix},$$

$$A_2Z_t = \begin{bmatrix} 0 & -(1-\alpha) & 0 & -1 & 0 \\ a_2 & 0 & 0 & 0 & a_1 \\ -a_4 & 0 & -1 & 0 & a_4 \end{bmatrix} \begin{bmatrix} \tilde{c}_t \\ \tilde{k}_t \\ \tilde{I}_t \\ \tilde{V}_t \\ \tilde{\mu}_t \end{bmatrix},$$

$$A_3X_{t+1} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & a_{12} & a_{10} \\ a_{18} & a_{17} & a_{16} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \tilde{y}_{t+1} \\ \tilde{L}_{t+1} \\ \tilde{m}_{t+1} \end{bmatrix},$$

$$A_4Z_{t+1} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ a_9 & 0 & 0 & 0 & a_{11} \\ a_{15} & -a_{18} & 0 & 0 & a_{16} \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \tilde{c}_{t+1} \\ \tilde{k}_{t+1} \\ \tilde{I}_{t+1} \\ \tilde{V}_{t+1} \\ \tilde{\mu}_{t+1} \end{bmatrix},$$

$$A_5X_t = \begin{bmatrix} a_8 & 0 & 0 \\ 0 & -a_{12} & -a_{10} \\ 0 & -a_{17} & -a_{16} \\ 0 & 0 & 0 \\ a_{19} & a_{20} & a_{21} \end{bmatrix} \begin{bmatrix} \tilde{y}_t \\ \tilde{L}_t \\ \tilde{m}_t \end{bmatrix},$$

$$A_6Z_t = \begin{bmatrix} a_6 & a_7 & 0 & 0 & 0 \\ -a_9 & 0 & a_{14} & 0 & a_{13} \\ -a_{15} & 0 & 0 & 0 & -a_{16} \\ 0 & 0 & 0 & -\zeta & 0 \\ a_{22} & a_{23} & a_{24} & a_{25} & a_{26} \end{bmatrix} \begin{bmatrix} \tilde{c}_t \\ \tilde{k}_t \\ \tilde{I}_t \\ \tilde{V}_t \\ \tilde{\mu}_t \end{bmatrix},$$

$$A_7U_t+_1 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} u^1_{t+1} \\ u^2_{t+1} \end{bmatrix},$$

$$A_8\varepsilon^X_{t+1} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -a_{12} & -a_{10} \\ -a_{18} & -a_{17} & -a_{16} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \varepsilon^y_{t+1} \\ \varepsilon^L_{t+1} \\ \varepsilon^m_{t+1} \end{bmatrix},$$

$$A_9\varepsilon^Z_{t+1} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ -a_9 & 0 & 0 & 0 & -a_{11} \\ -a_{15} & a_{18} & 0 & 0 & -a_{16} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \varepsilon^c_{t+1} \\ \varepsilon^k_{t+1} \\ \varepsilon^I_{t+1} \\ \varepsilon^V_{t+1} \\ \varepsilon^\mu_{t+1} \end{bmatrix}$$

Defining the Matrices J_4 and J_5

The equations (60)–(65) in matrix form can be written as

$$H_1\varepsilon^Z_{t+1} + H_2\varepsilon^X_{t+1} = H_3u_{t+1} + H_4e_{t+1} . \tag{a13}$$

Each of these terms is written explicitly below:

$$H_1\varepsilon^Z_{t+1} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ a_9 & 0 & 0 & 0 & a_{11} \\ a_{15} & a_{18} & 0 & 0 & a_{16} \end{bmatrix} \begin{bmatrix} \varepsilon^c_{t+1} \\ \varepsilon^k_{t+1} \\ \varepsilon^I_{t+1} \\ \varepsilon^V_{t+1} \\ \varepsilon^\mu_{t+1} \end{bmatrix}$$

$$H_2\varepsilon^X_{t+1} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & a_{12} & a_{10} \\ a_{18} & a_{17} & a_{16} \end{bmatrix} \begin{bmatrix} \varepsilon^y_{t+1} \\ \varepsilon^L_{t+1} \\ \varepsilon^m_{t+1} \end{bmatrix}$$

$$H_3U_{t+1} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u^1_{t+1} \\ u^2_{t+1} \end{bmatrix}, \qquad H_4e_{t+1} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} e^1_{t+1} \\ e^2_{t+1} \end{bmatrix} .$$

It follows that

$$\varepsilon^Z_{t+1} = H_5U_{t+1} + H_6e_{t+1} , \tag{a14}$$

where H_5 and H_6 are defined below:

$$H_5 = (H_1 + H_2M)^{-1}H_3 , \tag{a15}$$

$$H_6 = (H_1 + H_2M)^{-1}H_4 . \tag{a16}$$

Replacing (a14) in (75) one arrives at:

$$Z_{t+1} = J_1Z_t + J_4u_{t+1} + J_5e_{t+1} \tag{a17}$$

which is equation (83). The matrices J_4 and J_5 are given by equations (a18) and (a19):

$$J_4 = J_2 + J_3H_5 , \tag{a18}$$

$$J_5 = J_3H_6 . \tag{a19}$$

Definition of the Matrices D_1, D_2, D_3, and D_4

This section defines the matrices D_1-D_4 and derives the relationship between the VCV matrix of the residuals of a VAR on the variables Y, i, μ and M/P and the VCV matrix of the underlying disturbances vector $\{u, e\}$. First define the matrices D_1 and D_2 as follows. These matrices select the variables for the VAR according to the linear equation:

$$s_t = D_1 Z_t + D_2 X_t \,. \tag{a20}$$

$$D_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \qquad D_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

But since X is a linear function of Z we can use the definition of the matrix M from equation (73) to write s_t as a function solely of Z_t:

$$s_t = D_3 Z_t \,, \tag{a21}$$

where

$$D_3 = D_1 + D_2 M \,.$$

Define the VCV matrix of $\{u, e\}$ as Ω. The innovations in Z are equal to:

$$Z_t - E[Z_t] = J_6 \begin{bmatrix} u_t \\ e_t \end{bmatrix} \tag{a22}$$

where the matrix J_6 is the 4×4 matrix obtained by concatenating the matrices J_4 and J_5 given in (a18) and (a19). Using this notation, the innovations in s_t can be written as a linear function of the disturbances $\{u, e\}$:

$$s_t - E[s_t] = D_3(Z_t - E[Z_t]) = D_3 J_6 \begin{bmatrix} u_t \\ e_t \end{bmatrix} . \tag{a23}$$

The matrix D_4 referred to in equation (94) is then defined as $D_4 = D_3 J_6$.

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