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Deficits and Cycles*

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This paper shows that periodic equilibria may arise in a simple overlapping generations model with capital. If the government follows a policy of fixing the value of the deficit, rather than fixing the value of government debt, then the difference equation that describes competitive equilibria may posses complex roots in the neighborhood of the golden rule stationary state. One may show that if there exist parametric families of economies for which these roots change stability then, locally, there exists an invariant closed curve. The paper provides two simple examples that generate such equilibria, and it solves these examples numerically. *Journal of Economic Literature* Classification Numbers: 021, 022, 023, 131.

1. INTRODUCTION

This paper examines periodic equilibria that may arise in Diamond's [6] overlapping generations (O.G.) model with capital. The existence of cycles is established using the Hopf bifurcation theorem (see Guckenheimer and Holmes [11]) and two examples are presented that arise from simple standard utility and production functions.

A number of related papers have recently appeared in the literature. Azariadis [1], and Azariadis and Guesnerie [2] study cycles in a one good O.G. model and Grandmont [10] extensively investigates the existence of periodic equilibria in this model by applying the theory of "Flip" bifurcations.

The Flip bifurcation is less robust than the Hopf bifurcation since it disappears as the period length is shortened and for this reason one might be interested in establishing that a Hopf bifurcation can also occur in com-

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petitive economic models. There is another motive for establishing that a variety of economic assumptions may lead to periodic equilibria, since any particular example can be criticized as being "unlikely" or requiring "implausible assumptions." For example, some critics might argue that Grandmont's example is uninteresting because it requires the assumption that savings are a decreasing function of the rate of interest. The mechanism that drives the Hopf bifurcation which we describe does not rely on this assumption, and it provides a further example of periodic equilibria within a class of models that is widely used in macroeconomics.

Other authors have used the Hopf bifurcation theorem in related work. Benhabib and Nishimura [4, 5] present similar results in an infinite horizon continuous time model with multiple capital goods. Benhabib [3] generates a discrete time Hopf bifurcation in an O.G. model with an active monetary feedback rule and Reichlin [15] presents an example with a Leontieff technology and a variable labor supply. In contrast to the examples in [4, 5] and [15] this paper deals with cycles that arise only if the government pursues a particular policy. The motive for a government sector is not discussed in the paper although the examples do suggest that the way in which government expenditure is financed may have an important effect on the nature of a competitive equilibrium.

2. Model Structure

The model is a two-period lived overlapping generations economy with capital. Agents supply labor inelastically, when young, to a constant-returns-to-scale neoclassical production sector and consume the interest plus principle on their investments in old age. This structure allows one to derive asset demand equations by the young (see Diamond [6] for details) of the form

$$A_{t+1} = A(\omega_t, R_{t+1}). \tag{1}$$

The term ω_t represents the real wage and R_{t+1} is the gross real rate of return on assets held between periods t and $t+1$. Competition in production implies that the demand for capital by firms is a function of the interest factor; $k_{t+1} = k(R_{t+1})$. Similarly one may derive an expression for the real wage, $\omega_t = \phi(R_t)$, that is referred to as the factor price frontier. In the sequel, the following assumptions will be imposed on the functions A, ϕ, and k.

(AI) A, ϕ, and k are of class C^k for $k \geqslant 6$.

(AII) $A_1 > 0$.

(AIII) $\phi' < 0, k' \leqslant 0$.

(AI) is necessary to apply bifurcation theory to sequences of competitive equilibria. (AII) follows from the normality of current consumption and conditions (AIII) follow from concavity of the production technology.

It is assumed that all private assets are held by the young either as capital k_t or in the form of government debt B_t. Asset market equilibrium requires that

$$B(R_t, R_{t+1}) = B_t, \tag{2}$$

where

$$B(R_t, R_{t+1}) \equiv A(\phi(R_t), R_{t+1}) - k(R_{t+1}). \tag{3}$$

The function B represents the *net* private indebtedness of the government to the public. It is important to recognize that B_t may be positive or negative. If B_t is negative then the net private ownership of capital is less than the economy's stock of capital. This situation may be sustained by a government policy of purchasing shares in private corporations and issuing zero government debt. If one assumes that public production is a perfect substitute for private production then one may identify a negative value of B_t with a world in which the government undertakes production activity and issues positive debt of a lower value than the value of the public capital stock. If one adopts this latter interpretation then B_t represents government debt minus the value of public capital and the assumption of negative B_t accords with recent estimates of Eisner and Pieper [7] for the U.S. economy, that is, the net worth of the U.S. government in recent years has been *positive*!

3. Characterizing Equilibria

This paper departs from Diamond [6] in the description of policies followed by the government. Diamond assumes that $B_t = \bar{B}$ for all t which implies that competitive equilibria may be described as sequences of interest factors that are generated by a first order difference equation.[1] Instead, it is assumed that the government follows a policy of maintaining a constant zero budget deficit,[2] which, from the government budget constraint, implies that

$$B_t = R_{t-1} B_{t-1}. \tag{4}$$

[1] A Hopf bifurcation cannot occur in the case studied by Diamond since the policy of fixing $B_t = \bar{B}$ induces only first order dynamics which cannot possess complex eigenvalues.

[2] The policy of a fixed deficit is studied in [13].

Balanced budget policies induce competitive equilibria that are described as solutions to the difference equation

$$B(R_t, R_{t+1}) - R_t B(R_{t-1}, R_t) = 0 \tag{5}$$

with initial conditions $k(R_0) = k_0$, $B(R_0, R_1) = R_0 B_0$. Stationary equilibria must obey the relationship, $B(R^*, R^*)\ (1 - R^*) = 0$, from which it follows that a stationary equilibrium is either balanced, $B(R^*, R^*) = 0$, or golden rule, $R^* = 1$ (see Gale [9] for an elaboration of these definitions).

A more interesting class of equilibria is represented by periodic solutions to Eq. (5). The existence of such equilibria, in the neighborhood of a stationary state (R^*, R^*), may be established by applying the theory of a Hopf bifurcation. Heuristically, if one imposes the regularity condition $B_2(R^*, R^*) \neq 0$, one can solve (5) for R_{t+1} and thus generate a second order nonlinear difference equation in a neighborhood of the stationary state, of the form $(R_{t+1}, R_t) = F(R_t, R_{t-1})$. The eigenvalues of the Jacobian matrix DF evaluated at the stationary state are the solutions of the equation

$$\lambda^2 - \lambda\left(\frac{B}{B_2} + R^* - \frac{B_1}{B_2}\right) - R^* \frac{B_1}{B_2} = 0.$$

Suppose now that the characteristics of the economy are indexed by a parameter μ. A Hopf bifurcation occurs at, say, $\mu = \mu_0$ if the corresponding eigenvalues $\lambda_1(\mu)$, $\lambda_2(\mu)$ are complex conjugates and cross the unit circle at μ_0.

It is immediate to see that this phenomenon cannot occur at a balanced stationary state, $B(R^*, R^*) = 0$ since the associated eigenvalues are then $\lambda_1 = R^*$, $\lambda_2 = B_1/B_2$, which are both real. For this reason attention will be focused on the golden rule stationary state (1, 1). More precisely, let the characteristics of the economy (A_μ, ϕ_μ, k_μ) be indexed by a real parameter μ in some open interval of μ_0. The following propositions establish conditions under which a Hopf bifurcation occurs in a neighborhood of the golden rule stationary state at the critical parameter value μ_0. To guarantee the existence of a well-defined dynamical system in the neighborhood of the golden rule one requires the following regularity assumption:

(AIV) $B_2(1, 1, \mu) \neq 0$.

To apply the Hopf bifurcation theorem one also requires[3]:

(AV) $B(1, 1, \mu)$ is twice continuously differentiable with respect to μ.

[3] It is strictly only necessary to impose (AIV) and (AV) for μ in the neighborhood of a bifurcation point, μ_0 (see Theorem 1).

PROPOSITION 1. Assume (AI), (AIV), (AV). Then there exists an open neighborhood $U \subset \mathbb{R}^2$ of (1, 1) and a one parameter family of mappings F_μ of U into $\mathbb{R}^2$ such that the local dynamics of competitive equilibria are completely described by the map $F_\mu\colon (x, y) \to (G(x, y; \mu), x)$. The function G is of class C^k, $k \geqslant 6$, and is implicitly defined by the equation

$$B(x, G(x, y; \mu); \mu) - xB(x, y; \mu) = 0. \tag{6}$$

The family of mappings F_μ is twice continuously differentiable in μ.

The proof of Proposition 1 follows directly from the implicit function theorem.

Stationary equilibria are represented by fixed points of F_μ. The local behavior of nonstationary equilibria is governed by the roots of DF_μ in the neighborhood of a fixed point. The following theorem establishes local conditions for the existence of an invariant closed curve.

THEOREM 1. (Hopf Bifurcation Theorem).[4] *Let $F_\mu(x)$ be a one-parameter family of C^k mappings, $k \geqslant 6$, from some open subset U of $\mathbb{R}^2$ into $\mathbb{R}^2$. Let $x(\mu_0) \subset U$ be a fixed point of F_μ at which the eigenvalues of DF_μ are complex conjugates $\lambda(\mu_0)$, $\bar{\lambda}(\mu_0)$. Let F be C^2 in μ. Assume:*

(A) $|\lambda(\mu_0)| = 1$ *but* $\lambda^j(\mu_0) \neq 1$ *for* $j = 1, 2, 3, 4$,

(B) $(d/d\mu)(|\lambda(\mu_0)|) = d > 0$.

Transform the map F_μ to polar coordinates and identify an open neighborhood V of (r_0, θ_0) where $x(\mu_0) = r_0 \cos\theta_0$, $y(\mu_0) = r_0 \sin\theta_0$ by setting $(x, y) \in U$ equal to $(r\cos\theta, r\sin\theta)$ for $(r, \theta) \in V$. Then there exists a C^{k-4} change of coordinates h so that the expression of $hF_\mu h^{-1}$ in polar coordinates has the form

$$hF_\mu h^{-1}(r, \theta) = (r(1 + d(\mu - \mu_0) + ar^2), \theta + c + br^2) + \text{higher order terms.}$$

If $k \geqslant 6$ and $a < 0$ $(a > 0)$ then there exists a right (left) neighborhood of μ_0 in which there is an invariant attracting (repelling) closed curve for the map F_μ in V.

In Section 4 this theorem is applied to two simple examples. Proposition 1 establishes that competitive equilibria in the neighborhood of a stationary state are described by a C^k map. To establish the existence of an invariant closed curve, it remains to show that the eigenvalues of $DF_\mu|^*_{R=1}$ may be complex and that there exists a critical value of the bifurcation parameter μ_0 for which these roots change stability. One must also check

[4] Theorem 1 is adapted from Iooss [12] and Guckenheimer and Holmes [11] and the proof follows directly from restricting the treatment in [12] to consider only locally defined maps.

that conditions (A) and (B) are satisfied. The following proposition narrows down the class of economies in which one may expect to observe a Hopf bifurcation.

PROPOSITION 2. *Assume* (AI), (AII), (AIII), (AIV). *Let* λ^a, λ^b *be the roots of* DF_μ *evaluated at* (1, 1). *Then* λ^a, λ^b *are complex conjugates only if*

(i) $B_2 > 0$,
(ii) $B(1, 1) < 0$.

Proof. The characteristic polynomial of DF_μ evaluated at (1, 1) is given by,

$$P(\lambda) \equiv \lambda^2 - \lambda\left(\frac{B}{B_2} + 1 - \frac{B_1}{B_2}\right) - \frac{B_1}{B_2} = 0.$$

$P(\lambda)$ has complex roots only if

$$d \equiv \left(\frac{B}{B_2} + 1 - \frac{B_1}{B_2}\right)^2 + \frac{4B_1}{B_2} < 0.$$

Rearranging terms one may show that,

$$d = \left(\frac{B}{B_2} + 1 + \frac{B_1}{B_2}\right)^2 - \frac{4BB_1}{(B_2)^2},$$

hence,

$$d < 0 \Rightarrow B < \left(\frac{B}{B_2} + 1 + \frac{B_1}{B_2}\right)^2 \frac{(B_2)^2}{4B_1} < 0,$$

where the final inequality follows since $B_1 < 0$ from (AII), (AIII) and $B_2 \neq 0$ from (AIV). Regrouping terms one may also write, $d = ((B - B_1)/B_2)^2 + 1 + 2(B + B_1)/B_2$, and since $B_1 < 0$ by (AII), (AIII), and $B < 0$ from above, $d < 0 \Rightarrow B_2 > 0$.

Proposition 2 establishes that a Hopf bifurcation may occur only in economies at which the private sector is a net debtor at the golden rule stationary state. One also requires $B_2 > 0$. Economic theory does not place strong restrictions on the functions B_1 and B_2 and it is not difficult to find families of economies for which the roots of DF_μ cross the unit circle to which one may directly apply Theorem 1. Two such examples are discussed below.

4. Some Simple Examples

The following two examples are generated from simple technologies and preferences. The first example contains an attracting invariant closed curve and the second contains one that is repelling.[5]

Example 1. Production is C.E.S. with 100% depreciation, i.e., $f(k_t) = (1/\alpha)(1 + \alpha k_t^{\gamma})^{1/\gamma}$, and utility is Cobb–Douglas with weight, s, on future consumption which implies: $A(R_t) = \phi(R_t)\, s$, where $0 < s < 1$. The competitive equilibria of this example are solutions to the difference equation $B(R_t, R_{t+1}) - R_t B(R_{t-1}, R_t) = 0$, and the map F_μ has the representation

$$x_{t+1} = \left[\left(\frac{s(1-\alpha x_t^{-d})^{-1/d}}{\alpha} - x_t s(1-\alpha\, y_t^{-d})^{-1/d} + x_t(x_t^d - \alpha)^{-(1+d)/d}\right)^{-d/(1+d)} + \alpha\right]^{1/d}, \tag{7}$$

$$y_{t+1} = x_t, \tag{8}$$

where $d \equiv \gamma/(1-\gamma)$ is the elasticity of substitution minus one.

For d in the range $(0, \infty)$ capital and labor are substitutes in production, in the range $(-1, 0)$ they are complements and for $d = 0$ the production technology collapses to Cobb–Douglas. One may establish that a sufficient condition for complex roots at a bifurcation point is given by

$$|s - \alpha + \alpha(1+d)| < 2\alpha(1+d) \tag{9}$$

and that these roots have unit length when

$$\frac{s(1-\alpha)}{1+d} = 1. \tag{10}$$

Fixing any two of these parameters such that (9) holds one may choose the

[5] Theoretically one may compute the value of the parameter, a, referred to in Theorem 1 and demonstrate analytically whether a particular example is attracting, $a < 0$, or repelling, $a > 0$. Formulae for the computation of, a, and for the period and phase of the invariant circle are presented in Iooss [12]. In practise, however, the computations involved are very lengthy and the expressions depend in a complicated way on third derivatives of utility functions and of production technologies. Since economic theory does not place restrictions on third derivatives one might expect that either repelling or attracting cycles may exist. Both possibilities are of interest in the light of the results of Fuchs [8] and Grandmont [10] which suggests that a learning mechanism may reverse the stability of the dynamics of perfect foresight equilibrium models.

third parameter to be the bifurcation index μ. A bifurcation occurs if, as μ varies, $s(1-\alpha)/(1+d)$ passes through unity.

Figure 1 represents a computer simulation of this example in the neighborhood of the golden rule stationary state. Figure 1 was generated for parameter values of $\alpha = (0.5)$, $(1+d) = (0.24)$ and $s = (0.5)$. Holding $\alpha = (0.5)$, $s = (0.5)$, a bifurcation occurs as $(1+d)$ passes through (0.25). If $(1+d)$ is lowered towards zero then this example continues to generate a stable attracting closed curve until $(1+d)$ reaches (0.15). At this point the curve breaks, and repeated iterates of the map F_μ cause the trajectory of (x_t, y_t) to move outside the domain on which $G(x, y)$ is defined.

EXAMPLE 2. In this example technology is Leontieff with 100% depreciation, which is a limiting case of the technology considered in example 1. The factor price frontier is given by $\omega_t = \alpha - R_t$ and the demand for capital is interest inelastic and normalized to unity, $k_{t+1} = 1$.

Since the function, $B(R_t, R_{t+1}) \equiv A(\phi(R_t), R_{t+1}) - k(R_{t+1})$, must have a positive derivative with respect to R_{t+1} in order to generate complex roots it is assumed that, $A_2 > 0$, unlike example 1 in which the asset demand function depended only on the wage. Specifically the function A is assumed to take the form; $A = s\omega_t R_{t+1}$, which may be derived from the indirect

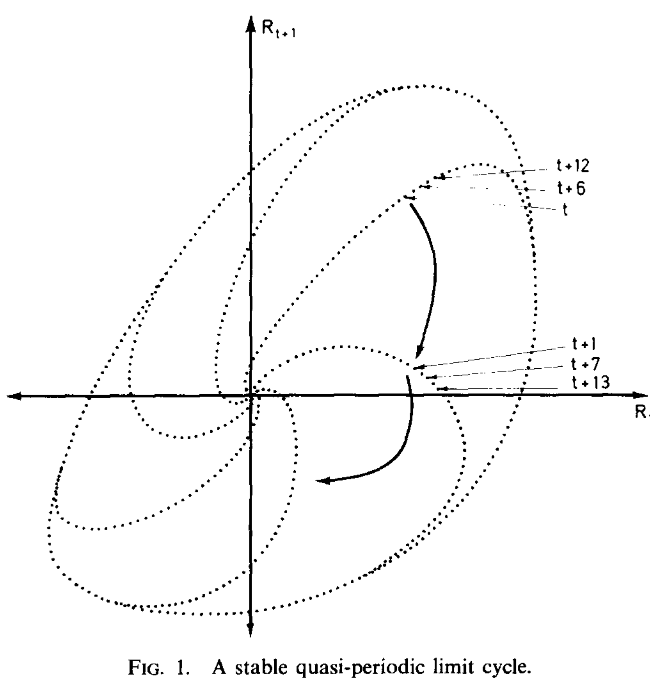

FIG. 1. A stable quasi-periodic limit cycle.

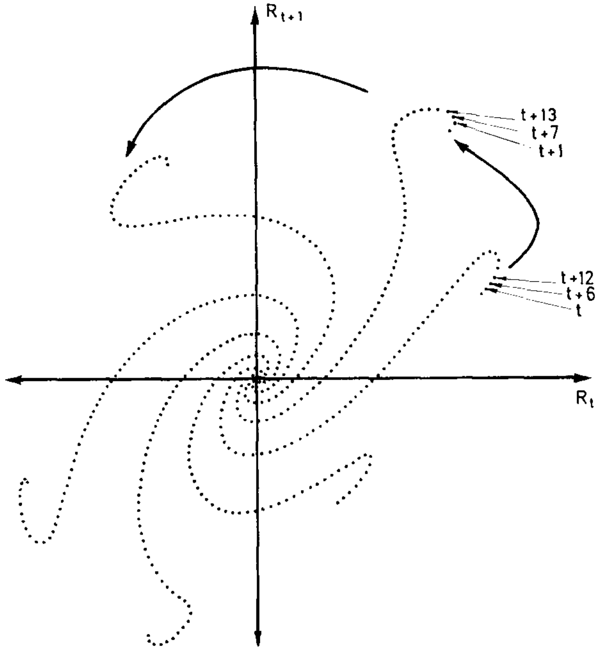

FIG. 2. An unstable strictly periodic limit cycle.

utility function $V=\omega_t \exp(sR_{t+1})$ by applying Roy's identity.[6] The map F_μ is given by

$$x_{t+1}=\frac{1+x_t^2\, s(\alpha-y_t)-x_t}{s(\alpha-x_t)}, \tag{11}$$

$$y_{t+1}=x_t, \tag{12}$$

and the roots of $DF_\mu|_{R=1}^{*}$ are complex if

$$\left[1+\frac{s(\alpha-1)-1}{s(\alpha-1)}+\frac{1}{\alpha-1}\right]^2<\frac{4}{\alpha-1}. \tag{13}$$

Furthermore, these roots change stability as

$$\alpha \gtrless 2.$$

Figure 2 represents a computer simulation of this example for parameter values of $\alpha=2.01$, $s=0.5$. The example differs from Fig. 1 in two respects.

[6] V is a proper indirect utility function for $0<R_{t+1}<1/s$. See [14, p. 89] for a definition of the properties of indirect utility functions.

First, the periodic equilibrium is unstable in Example 2 and stable in Example 1. Second, Fig. 2 depects a strictly periodic equilibrium of period 6 whereas Fig. 1 is an example of a quasi-periodic equilibrium in which the system "almost" repeats itself every six periods but no state reoccurs exactly. Both types of behavior can occur theoretically for open sets of parameter values. As s moves towards zero, in example 2, the stable six cycle breaks into a quasi-periodic equilibrium around $s=0.498$. If s is further lowered, cycles of different periods appear until at a value of $0.25<s<0.3$, all trajectories appear to be captured by a stable four cycle.

The economics behind the possibility of complex roots rests on the fact that the interest rates at dates t and $t+1$ may pull the private demand for government bonds in different directions. An increase in R_t tends to lower the demand for assets, since it depresses the current wage, and an increase in R_{t+1} has the opposite effect since it lowers the private capital stock and may increase private savings. The following explanation deals only with the case in which $B_2>0$, although periodic equilibria may also occur (for different reasons) if $B_2<0$.[7]

It is helpful to consider two possible cases. Suppose first that $B(1, 1)>0$, and consider a competitive sequence of interest factors $\{R_t\}_{t=2}^{\infty}$ where $R_0=1$, $R_1>1$. It follows from the government budget constraint that $B(R_1, R_2)=R_1 B(R_0, R_1)$. If $R_1>R_0=1$ then the government must float enough debt in period 2 to cover the principle *plus* interest on the debt outstanding in period 1, i.e., $B(R_1, R_2)>B(R_0, R_1)$.

One may approximate the growth in B between periods t and $t+1$ by the expression $\Delta_t=B_1(R_t-R_{t-1})+B_2(R_{t+1}-R_t)$. The first term in this expression represents the downward pull of the current interest factor on private saving through its effect on the wage and the second term represents the upward pull of the future interest rate. In period one, the first term of Δ_1 is unambiguously negative since $R_1>1$ and $B_1<0$. It follows that the interest factor in period 2 must be strictly greater than the interest factor in period 1, $(R_2>R_1)$, to induce the private sector to hold the increased quantity of debt. This same argument may be applied in period 3 to show that $R_3>R_2$ and, by induction, one may show that the competitive interest rate sequence must increase monotonically in the neighborhood of the golden rule and hence cycles cannot occur if $B(1, 1)>0$.

Suppose, alternatively, that the private sector is a net debtor at the golden rule, i.e., $B(1, 1)<0$. Consider, once again, the competitive sequence $\{R_t\}_{t=2}^{\infty}$ for $R_0=1$, $R_1>1$. In this case the government must reinvest the interest on its assets in period 2 in order to maintain a zero deficit. It

[7] In the case $B_2<0$ a Flip bifurcation may occur for reasons similar to those discussed by Grandmont [10].

follows from the government budget constraint that $B(R_1, R_2) = R_1 B(R_0, R_1)$. But since B is negative and, $R_1 > 1$, net private holdings of government assets must be a larger negative number in period one than in period zero, i.e., $\Delta_1 = B_1(R_1 - 1) + B_2(R_2 - R_1) < 0$. Once again the first term in this expression is unambiguously negative but it follows that the sign of $R_2 - R_1$ is ambiguous. In a periodic equilibrium the effect of a rise in the current interest factor, $B_1(R_1 - 1)$, may cause the net demand for assets to fall more than is required to maintain equilibrium. To restore balance, the effect of the future interest factor must be positive, $R_2 > R_1$, and so initially the equilibrium interest rate sequence begins to rise. But as R_t increases, the amount by which net government assets must increase each period accelerates, i.e., Δ_t becomes a larger negative number and the effect of the current interest rate, $B_1(R_t - R_{t-1})$, is no longer sufficient to increase private indebtedness by enough to maintain equilibrium. At this point the effect of the future interest factor, $B_2(R_{t+1} - R_t)$, must also become negative, $R_{t+1} < R_t$, and the sequence of competitive interest factors begins to fall. This process generates a cyclic equilibrium in the neighborhood of the golden rule, which may be locally either convergent or divergent.

A bifurcation occurs in a family of economies as the stability of the equilibrating process changes. In Example 1,as the elasticity of substitution $(1 + d)$ tends to zero, the effect of the future interest factor on the net private demand for government assets becomes smaller. The value of B_2 is given by $(1 + d)(1 - \alpha)^{(-(1 + 2d)/d)}$ which tends to zero as $(1 + d)$ tends to zero. As $(1 + d)$ becomes smaller the amount by which R_{t+1} must adjust to maintain a given increase in the demand for assets becomes increasingly larger. A bifurcation occurs when the required response becomes so great that the sequence of interest factors that will maintain equilibrium moves from a locally convergent to a locally divergent sequence. A similar process occurs in example 2 but in this case the effect of the future interest rate on net asset demand is given by $B_2 = s(\alpha - 1)$. In this example it is the parameter α which generates a bifurcation but in both cases the bifurcation occurs as a process that is inherently cyclical changes stability.

5. Concluding Comments

One reason for studying nonlinear cycles is to develop an endogenous theory of the business cycle that can compete with the existing dominant paradigm; that is, the linear stochastic model. A variety of economic mechanisms may be responsible for generating periodic equilibria endogenously; it is hoped that the above examples may stimulate further research in this area.

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