Fiscal Policy, Equity Premia and Heterogeneous Agents

Preliminary Version

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Abstract

I construct a model of long-lived agents that provides a tractable generalization of the consumption-based asset pricing model in which the set of agents that are active in the aset markets changes over time. As a consequence of this assumption, the pricing kernel depends not only on aggregate consumption in consecutive periods, but also on aggregate wealth. I calibrate the model to fit the data from 1950 to 1979 and I show that the calibrated model does a relatively good job at explaining the observed data on debt, on the value of the stock market and on the rates of return on safe and risky assets.

1 Introduction

In the period from 1950 through 1979 there were only three years when the interest rate on three month treasury bills exceeded the rate of growth of nominal GDP. After 1979 the situation was very different. Figure 1 displays these data.

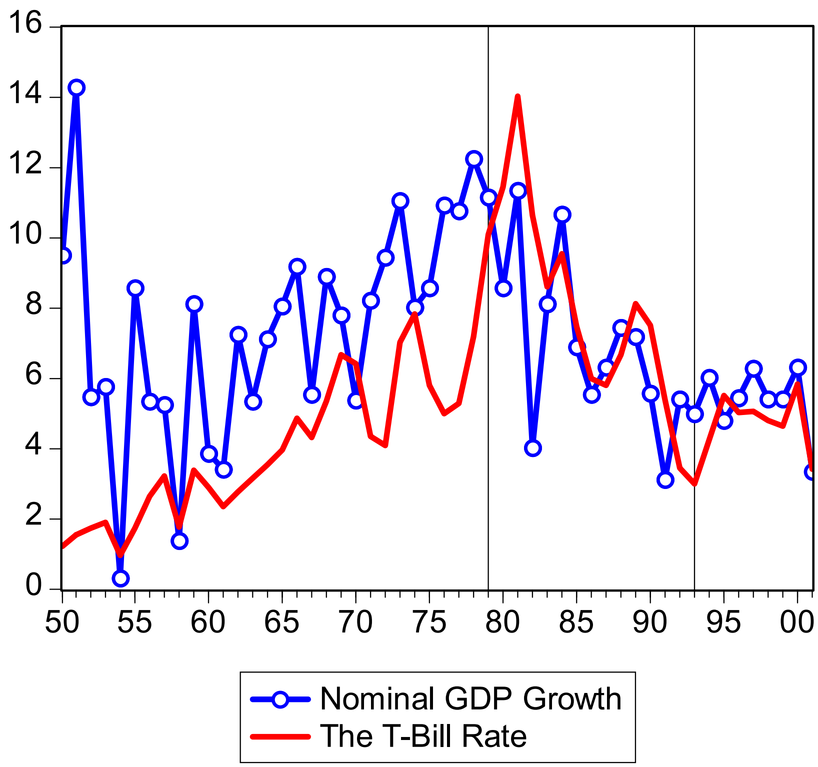

Figure 1

In 1950 the stock of government debt was equal to 87% of GDP. The debt to GDP ratio declined slowly during the period from 1950 to 1979 and by the end of this period it had reached 37%. Most of the decline occurred as a result of a high rate of growth in the tax base: The ratio of the nominal interest rate to the nominal growth rate over this period was equal on average to 0.97. The government also ran a small primary surplus equal on average to 1.2% of GDP.

In 1979 there were two important events that contributed to change the situation. In the third quarter of 1979 Paul Volcker took over from William

Miller as Chairman of the Federal Reserve and at this time there was a dramatic shift in Fed policy that led to a steep increase in nominal interest rates. At about the same time government spending increased and the rate of income tax fell. The consequence for the government budget is illustrated in Figure 2.

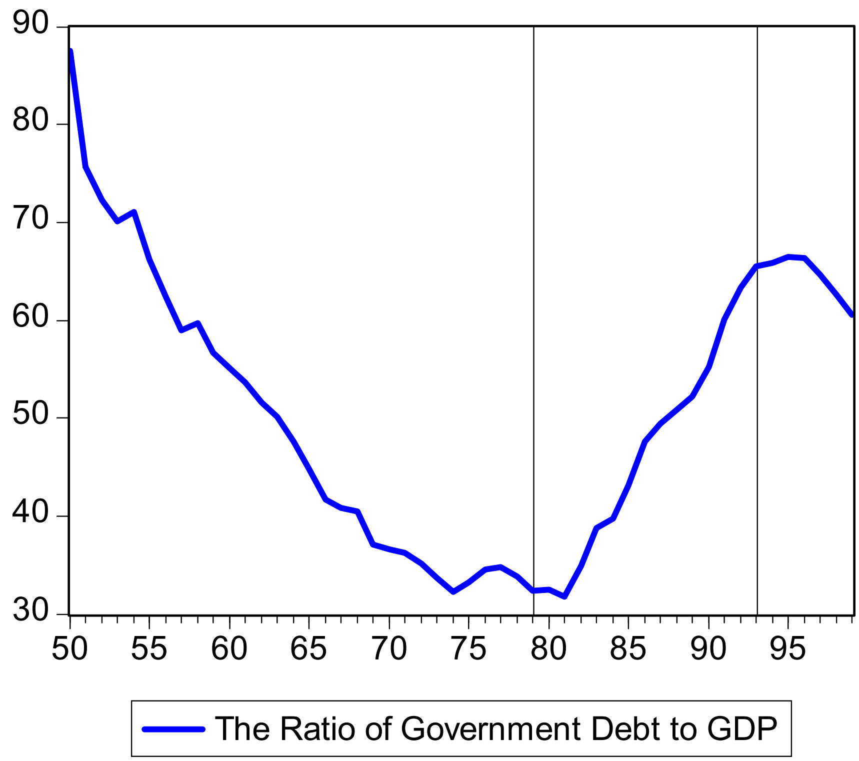

Figure 2

After 1979, the ratio of debt to GDP increased steadily to a new peak in 1996 of 66% of GDP. In 1993, congress passed the deficit reduction act, and by 1996 the effect of this act can be seen on the debt to GDP ratio which began once again to decline.

During the period since 1950 there have also been dramatic changes in the equity markets. Figure 3 shows that the price earnings ratio picked up in 1979 and began a period of gains that continued until the recession of 2002. In this paper I will argue that the changes in the relationship between the interest rate and the growth rate documented in Figure 1, the changes in the debt to GDP ratio documented in Figure 2, and the growth in the post 1979 value of the market depicted in Figure 3, can be well explained by an asset pricing model in which the set of agents changes over time.

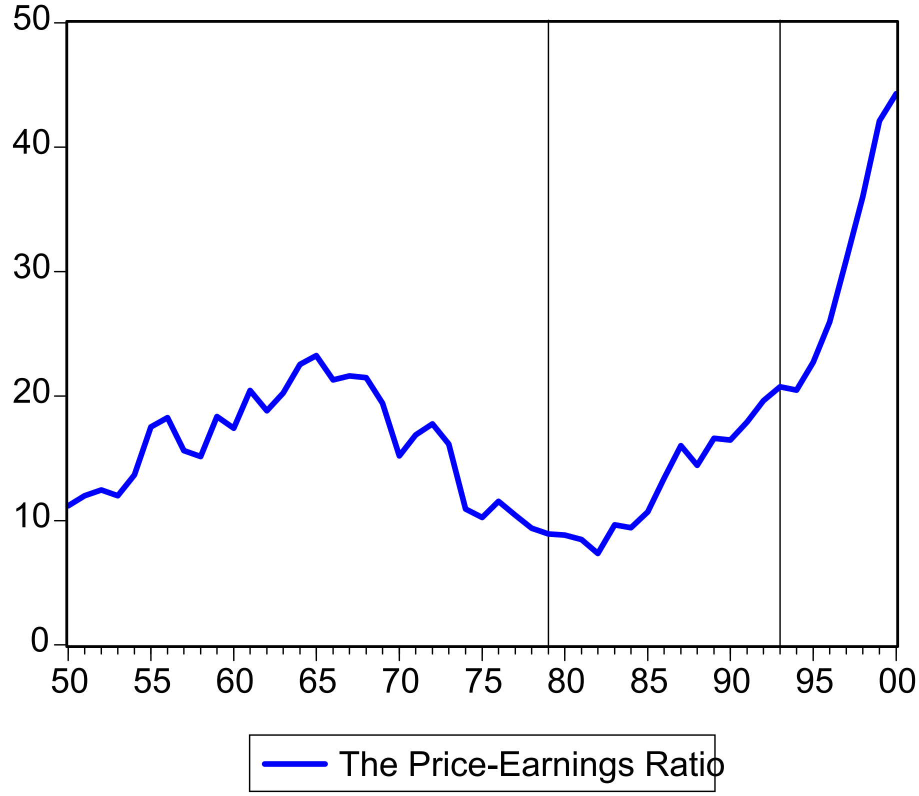

Figure 3

I will construct a model that can account for all of the features of the data described above. I calibrate my model in a way that implies that the government is able to borrow at an interest rate lower than the rate of growth of tax revenues. In economic terms, it runs a Ponzi scheme. Although it is widely recognized that the safe rate of return is low, there is a paucity of formal models that capture this fact. The standard intertemporal model of a single infinitely lived household cannot explain how the interest rate could be less than the growth rate without appealing to an unreasonably high coefficient of risk aversion. The problem of formulating a consistent economic model of these data is compounded by the fact that historically the return to the stock market has been higher than the return to T-bills by about 5% points. Figure 4 illustrates these two rates of return for the period under consideration.

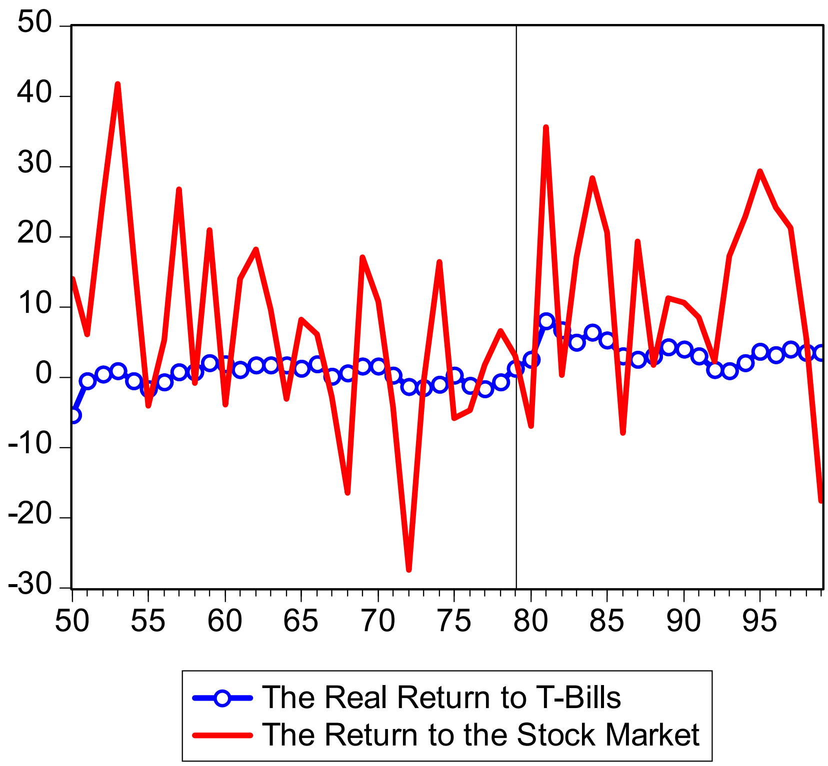

Figure 4

The growth rate of earnings is reasonably well described by a random walk. In a representative agent economy, if agents have isoelastic preferences, the price-earnings ratio should be constant. Figure 3 illustrates that in the data, the price-earnings ratio displays considerable low frequency movements. An associated puzzle, first documented by Robert Shiller, is that of excess volatility. Since the value of equity is predicted to equal the discounted present value of future dividends, equity should be less volatile than dividends. In the data it is more volatile. To solve this puzzle, together with the associated problem of the high equity premium, my model drops the representative agent assumption. I allow, instead, for the set of participants in the asset markets to change over time. As a consequence, the pricing kernel (also called the stochastic discount factor) is no longer equal to a power function of consumption growth; instead, it depends not only on aggregate consumption in adjacent periods but also on the price of equity. I will calibrate the model to fit the facts of the equity premium and I will show that this calibration leads to a prediction for the time series properties of debt and of asset prices that is consistent with the broad features of the data in Figures 1 through 4.

2 Key Features of the Model

The model combines features of Lucas' [3] asset pricing model with Blanchard's [4] paper on long-lived agents. I assume that there is a unit measure of agents that actively participate in the asset markets. Each period a fraction $(1-\chi)$ of these agents (chosen at random) are replaced. The probability that any agent will survive into the subsequent period is equal to χ. Agents have logarithmic preferences defined over consumption and leisure. Each agent is endowed with two assets; a single unit of physical capital that depreciates at rate $(1-\delta_K)$ and a single unit of human capital that depreciates at rate $(1-\delta_H)$. Physical capital is tradeable, human capital is not. Output is produced by combining physical and human capital in a Cobb-Douglas technology, subject to an aggregate technology shock. The rate of growth of the technology shock is a random walk and hence aggregate GDP is stochastic. Agents are able to trade a complete set of Arrow securities, indexed to the shock, and they are able to trade life insurance in a perfect annuities market.

The main idea of the paper is that in this environment one can derive a simple asset pricing formula that generalizes the Lucas "tree model" to a world of heterogeneous agents. Specifically, let $Q_{t+1}(\gamma')$ represents the price of an Arrow security that pays one unit of consumption in state γ'. I will show that $Q_{t+1}(\gamma')$ is related to aggregate consumption growth and aggregate wealth by the expression

$$Q_{t+1}(\gamma) = \frac{aC_t}{bC_{t+1} + cW_{t+1}}$$

where a, b and c are functions of the parameters of the model, C_t is aggregate consumption and W_{t+1} is aggregate wealth. In the special case of a single representative agent, a is the agent's discount factor, $b = 1$ and $c = 0$. This choice of parameters leads to the familiar consumption based asset pricing kernel.

More generally, the pricing kernel will itself depends on aggregate wealth. For example, let there be one unit of physical capital in aggregate that pays a dividend d_{t+1}, depreciates at rate δ_K and sells for price p_t. This unit of capital would be priced by the no-arbitrage relationship

$$p_t = d_t + E_t \left[\left(\frac{aC_t}{bC_{t+1} + cp_{t+1}} \right) \delta_K p_{t+1} \right]. \tag{1}$$

In an endowment economy, Equation (1) reduces to the following expression in the price dividend ratio $\hat{p}_t$.

$$\hat{p}_t = 1 + E_t \left[\left(\frac{a}{b + c\hat{p}_{t+1}} \right) \gamma_{t+1} \delta_K \hat{p}_{t+1} \right].$$

I will use the fact that the pricing kernel depends on aggregate wealth to construct a model in a which the safe rate of interest is less than the growth rate but the risky rate displays a substantial equity premium.

3 Preferences

My economy is populated by infinite horizon families that have logarithmic preferences over consumption and leisure and discount factor $\beta \in [0, 1)$. These agents survive from one period to the next with probability χ. They maximize the discounted expected present value of a function,

$$U_t^s = E_t \left\{ \sum_{t=s}^{\infty} (\beta\chi)^{t-s} \left[(1-\lambda) \log (C_t^s) + \lambda \log (e_t^s - L_t^s) \right] \right\}$$

where C_t^s and $e_t^s - L_t^s$ represent consumption and leisure at date t of an agent born at date s. L_t^s represents time spent in market activities and e_t^s is the agent's time endowment measured in efficiency units. Each period, agents die with probability $(1 - \chi)$ and, importantly, the death probability is independent of age.

Agents trade a complete set of Arrow securities. The sequence of budget constraints faced by an agent is described below:

$$A_s^s = p_t K, \quad K = (1 - \delta_K), \tag{2}$$

$$\sum_{S'} \tilde{Q}_{t+1} (\gamma') A_{t+1}^s (\gamma') = A_t^s (\gamma) + \omega_t L_t^s + T_t^s - C_t^s, \quad t \geq s. \tag{3}$$

An agent born in period s supplies labor hours L_t^s to the market, he purchases consumption commodities C_t^s and accumulates Arrow securities $A_{t+1}^s(\gamma')$. There is a finite set of securities, one for each state γ'. A security pays one unit of consumption if state γ' is realized and zero otherwise. ω_t is the wage, L_t^s is hours worked, C_t^s is the household's purchase of consumption commodities and T_t^s represents lump-sum transfers. I assume that the wealth of an agent is finite at prices $\hat{Q}$ and that the value of consumption is less than or equal to the value of wealth.

Each period $(1-\chi)$ agents die (randomly selected from all existing generations) and they are replaced by a measure $(1-\chi)$ of new-born agents. Since these agents take identical decisions I treat them as a single agent and refer to them collectively as the new-born generation. The new-born generation is endowed with two sources of wealth. It owns $K = (1-\delta_K)$ units of physical capital (worth p_t units of consumption) of which a fraction δ_K survives into the subsequent period. It also owns $(1-\delta_H)$ efficiency units of labor in the period of birth and an endowment in each subsequent period that declines geometrically with a depreciation factor of δ_H. These assumptions imply that the economy as a whole is endowed with 1 unit of capital of which a fraction $(1-\delta_K)$ is owned by the new-born generation and a fraction δ_K is owned by the old. Similarly, the economy is endowed with a single unit of labor in efficiency units of which a fraction $(1-\delta_H)$ is owned by the new generation and a fraction δ_H by the old.

I define human wealth as follows,

$$H_t^s = e_t^s \omega_t + T_t^s + \sum_{S'} \tilde{Q}_{t+1} H_{t+1}^s.$$

Because every generation has the same survival probability, human wealth declines at the same rate for all generations. Hence there exists an aggregate concept of human wealth of which a fraction $(1-\delta_H)$ is owned by the new-borns and a fraction δ_H is owned by the old.[1]

Using this definition of human wealth one can represent the solution to the household's problem as follows:

$$C_t^s = \lambda (1-\beta\chi)(A_t^s + H_t^s), \tag{4}$$

$$\omega_t (e_t^s - L_t^s) = (1-\lambda)(1-\beta\chi)(A_t^s + H_t^s). \tag{5}$$

[1] To avoid unnecessary complications I assume that lum-sum transfers are allocated between young and old in proportion to the ratio of their ownership of human wealth.

Equations (4) and (5) are a consequence of the assumption of logarithmic preferences. They say that the agent allocates constant budget shares, as a fraction of wealth, to the purchase of consumption goods and leisure.

Following Blanchard [4] I assume the existence of a perfect annuities market. Agents pay the price $\tilde{Q}_{t+1}(\gamma')$ for a security that pays one unit of the consumption good if and only if the aggregate state γ' occurs *and* the agent survives into the subsequent period. In the event of the agent's death, his assets are returned to the life insurance company. Since I assume perfect competition in the annuities market, the price $\tilde{Q}_{t+1}(\gamma')$ paid by the agent will be related to the price $Q_{t+1}(\gamma')$ paid by the insurance company by the expression

$$\tilde{Q}_{t+1}(\gamma') = \chi Q_{t+1}(\gamma').$$

The individual receives a higher return on his assets than the return earned by the company. In return for this premium the company receives his assets when he dies. Since I assume the existence of a large number of agents, the life insurance company spreads its risks over many agents and it makes zero profits in aggregate. The premia received by the agents that survive are exactly covered by the assets of the agents who die.

4 Technology

I assume that output is produced with a Cobb-Douglas production function,

$$Y_t = S_t K^{1-a} L_t^{\alpha},$$

where K is the stock of capital, L_t is labor and S_t is a random productivity shock. Throughout the paper I use the notation that a variable without a superscript s is the integral of the corresponding subscripted variable over all generations; hence $L_t = \int L_t^s ds$ where s indexes generation. The aggregate stock of capital is normalized to 1 hence I can write the technology as a function of labor alone,

$$Y_t = S_t L_t^{\alpha}.$$

Productivity is assumed to follow a geometric random walk with drift

$$S_{t+1} = S_t \gamma_{t+1},$$

where γ_{t+1} is an error with mean $\gamma \geq 1$ that represents the innovation to the productivity shock. I assume that the support of γ_{t+1} is finite and that it corresponds to the set of Arrow securities in the sense that for each element of the support of γ_{t+1} there is an associated security.

Factor markets are competitive so that the real wage ω_t and the profit rate r_t are given by the expressions

$$\omega_t = \frac{\alpha Y_t}{L_t}, \quad r_t = (1-\alpha) Y_t, \tag{6}$$

where I have exploited the Cobb-Douglas functional form to write the marginal products of labor and capital in this way. Since preferences and technology are unit-elastic, the household will supply a fixed amount of labor in equilibrium. Combining Equations (4) and (5) with (6) and integrating over all agents gives

$$\frac{(1-\lambda) C_t}{\lambda (1-L_t)} = \omega_t = \frac{\alpha Y_t}{L_t}.$$

Since $Y_t = C_t$, it follows that labor supply is constant and equal to

$$L_t = \frac{\alpha\lambda}{(1-\lambda(1-\alpha))}. \tag{7}$$

By picking an appropriate choice for the units of measurement of labor hours, I am free to choose the units of measurement such that $Y_t = S_t$. In the subsequent discussion I will use this normalization and hence it will also be true that GDP growth Y_{t+1}/Y_t is equal to γ_{t+1}.

5 Government

A key focus of my paper is the effect of government debt and deficits on the asset markets. I assume that the government finances its spending with taxes and by issuing a dollar denominated asset B_t that pays a fixed nominal interest rate each period denoted r. Let the dollar price of this bond be q_t and let P_t be the dollar price of commodities. Then q_t is related to its own future value by the no arbitrage condition;

$$\frac{q_t}{P_t} = \sum_{\gamma'} Q_{t+1}(\gamma') \frac{(q_{t+1} + r)}{P_{t+1}}. \tag{8}$$

I assume that government finances its budget entirely by long bonds of this kind. As a consequence of this assumption, the government budget constraint each period is represented by the expression:

$$\frac{Y_{t+1}}{Y_t}\frac{(q_{t+1}+r)\,B_{t+1}}{Y_{t+1}P_{t+1}} = \left(\frac{(q_{t+1}+r)\,P_t}{q_t P_{t+1}}\right)\left(\frac{B_t\,(q_t+r)}{Y_t P_t}+\frac{D_t}{Y_t P_t}\right). \tag{9}$$

or using the symbols, $d_t = \frac{D_t}{Y_t P_t} \equiv d$ for the deficit to GDP ratio, $b_t = \frac{B_t}{P_t Y_t}$ for the debt to GDP ratio and $z_t = (q_t + r)\,b_t$ for the market value of debt plus interest we can write the government budget constraint as

$$\gamma_{t+1} z_{t+1} = \left(\frac{(q_{t+1}+r)\,P_t}{q_t P_{t+1}}\right)(z_t + d). \tag{10}$$

Using Equations (9) and (8) and the fact that $Y_{t+1}/Y_t \equiv \gamma_{t+1}$ we can write the no arbitrage condition as follows;[2]

$$1 = \sum_{\gamma'} Q_{t+1}(\gamma')\left(\frac{\gamma_{t+1} z_{t+1}}{z_t + d}\right). \tag{11}$$

It is an important feature of the data, that the debt/GDP ratio moves smoothly over time. But debt is measured as the book value at the date of issue and the market value of debt changes substantially from month to month as short term interest rates fluctuate. A 5% nominal coupon perpetuity will double in value if the nominal interest rate moves from 2% to 1%. This fact implies that a policy that sets the nominal interest rate can have a substantial effect on the values of outstanding assets. Interest rate policy will be an important feature of my model. The asset pricing Equation (11), in conjunction with a parallel equation that determines the value of equity, will be used to determine the market value of government debt, z_t. The market value is related to the book value, b_t, by the identity;

$$z_t \equiv b_t\,(q_t + r).$$

To determine the book value, b_t I will allow the Fed to pick the price q_t through open market operations in short-term securities. Since there is considerable evidence in the data that the price level moves slowly through time,

[2] I alow for both lump-sum taxes T and a tax on dividends of t. T is determined endogenously by the expression $T_t = \frac{D_t}{P_t} - t\,(1-\alpha)\,Y_t$.

I assume that the Fed picks policies that target the inflation rate and that it is able to set inflation equal to a constant rate of δ^{-1} :

$$\frac{P_t}{P_{t+1}} = \delta.$$

Using this assumption and rearranging terms in Equation (9), one can derive the following expression for the the evolution of the book value of the ratio of debt to GDP;

$$\gamma_{t+1} b_{t+1} = \delta \left(b_t \left(1 + \frac{r}{q_t} \right) + \frac{d}{q_t} \right).$$

In this expression r is the fixed dollar valued coupon on perpetuities, b_t is the book value of the debt to GDP ratio, γ_{t+1} is the growth factor of GDP, d is the deficit to GDP ratio and q_t is the dollar price of long bonds.

In an equilibrium it will be true that the value of all assets held by agents in the model is equal to the value of capital plus the value of government debt. This leads to the asset market equilibrium condition:

$$a_t = p_t + z_t.$$

By choosing q_t, the Fed determines what percentage of fluctuations in z_t are allocated to b_t and what percentage are allocated to q_t. The policy of picking the inflation rate equal to a constant implies that b_{t+1} is determined one period in advance; hence all deviations of z_t from its conditional expectation are met by changes in the price of long maturity government bonds.[3]

6 The Pricing Kernel

To complete the model I need to find an expression for the price of an Arrow security in terms of observable aggregate variables. Each agent alive at date t will choose consumption to obey the Equation (12), where $\pi(\gamma')$ is the probability that state γ' occurs,

$$\frac{1}{C_t^s} = \frac{\chi \beta \pi(\gamma')}{\chi Q_{t+1}(\gamma') C_{t+1}^s(\gamma')}. \tag{12}$$

[3]I have chosen this policy because it is relatively easy to solve and it leads to a smooth path for b_t. However, one could clearly allow for more complicated feedback rules in which q_t is chosen as a function of past inflation.

Rearranging this expression, and canceling χ from top and bottom, leads to the following linear equation that can be aggregated across agents:

$$Q_{t+1}(\gamma') C^s_{t+1}(\gamma') = C^s_t \beta \pi (\gamma'). \tag{13}$$

Let C^Y_t be the aggregate consumption of new-borns at date t and let C^O_t be the aggregate consumption of everyone else. Think of superscript Y as denoting "young" and superscript O as denoting "old". Using Equation (13) we can derive the following expressions for consumption of young and old;

$$C^Y_t = \lambda(1-\beta\chi)\left((1-\delta_H) H_t + (1-\delta_K) p_t\right) \tag{14}$$

$$C^O_t = \lambda(1-\beta\chi)\left(\delta_H H_t + \delta_K p_t + \frac{B_t}{P_t}(q_t + r)\right). \tag{15}$$

All agents consume a fraction $\lambda(1-\beta\chi)$ of their wealth. Young agents are endowed with $(1-\delta_H)$ units of human wealth and $(1-\delta_K)$ units of capital. Old agents are endowed with δ_H units of human wealth, δ_K units of capital and all of the outstanding stock of government debt.

We seek an expression for the security price $Q_{t+1}(S')$ in terms of the observable variables C_t, p_t and $\frac{B_t}{P_t}(q_t + r)$. To arrive at this expression observe first that Equation (13) will hold for all agents alive at dates t and $t+1$. This observation implies that

$$Q_{t+1}(\gamma') = \frac{\beta\chi\pi(S') C_t}{C^O_{t+1}(\gamma')}, \tag{16}$$

where the term χC_t in the numerator arises since a fraction χ of agents alive at date t will become old agents at date $t+1$. To derive an expression for C^O_{t+1} in terms of observable variables we need an expression for H_{t+1}. To derive this expression note that aggregate consumption and aggregate wealth are related by the expression,

$$C_{t+1} = \lambda(1-\beta\chi)\left(H_{t+1} + p_{t+1} + \frac{B_{t+1}}{P_{t+1}}(q_{t+1} + r)\right),$$

hence

$$H_{t+1} = \frac{C_{t+1}}{\lambda(1-\beta\chi)} - p_{t+1} - \frac{B_{t+1}}{P_{t+1}}(q_{t+1} + r). \tag{17}$$

And since

$$C^O_{t+1} = C_{t+1} - C^Y_{t+1}$$

we can use Equation (14) to write C^O_{t+1} as follows

$$C^O_{t+1} = C_{t+1} - \lambda (1 - \beta\chi) ((1 - \delta_H) H_{t+1} + (1 - \delta_K) p_{t+1}). \tag{18}$$

Putting together Equations (16) (17) and (18) gives the expression we seek

$$Q_{t+1}(\gamma') = \frac{\beta\pi(\gamma')\chi C_t}{\delta_H C_{t+1} + \lambda(1-\beta\chi)\left((\delta_K - \delta_H) p_{t+1} + (1-\delta_H)\frac{B_{t+1}}{P_{t+1}}(q_{t+1} + r)\right)}. \tag{19}$$

In the case when $\delta_K = \delta_H = \chi = 1$ this model collapses to a representative agent economy and Equation (19) becomes

$$Q_{t+1}(\gamma') = \frac{\beta\pi(\gamma') C_t}{C_{t+1}}$$

which is a standard expression for the pricing kernel in a representative agent economy with logarithmic preferences. More generally, asset prices will also depend on the value of aggregate wealth.

7 The Price of Capital

In this section I show how to value assets in an economy with no government for which $B_t = 0$ and $t = 0$. In this case all physical wealth is stored as capital. Absence of arbitrage implies that a unit of capital that depreciates at rate $(1 - \delta_K)$ and pays a dividend of $(1 - \alpha) Y_t$ will sell for price p_t at date t, where the price p_t is related to its own future value by the formula:

$$p_t = (1 - \alpha) Y_t + \sum \delta_K Q_{t+1}(\gamma') p_{t+1} \tag{20}$$

Since I have abstracted from the government sector, the pricing kernel Q_t is a function of aggregate consumption at consecutive dates and of the future value of capital;

$$Q_{t+1}(\gamma') = \frac{\beta\pi(\gamma')\chi C_t}{\delta_H C_{t+1} + \lambda(1-\beta\chi)(\delta_K - \delta_H) p_{t+1}}. \tag{21}$$

If we substitute Equation (20) into Equation (21) and let $\hat{p}_t$ stand for the ratio of p_t to $(1 - \alpha) Y_t$ one obtains the following recursion in the price/earnings ratio of a unit of capital

$$\hat{p}_t = 1 + E_t \left\{ \delta_K \left[\frac{\beta\chi}{\delta_H + \lambda(1-\beta\chi)(1-\alpha)(\delta_K - \delta_H)\hat{p}_{t+1}} \right] \hat{p}_{t+1} \right\}. \quad (22)$$

In the case in which $\delta_K = \delta_H = \chi = 1$, this formula collapses to the standard asset pricing relationship.

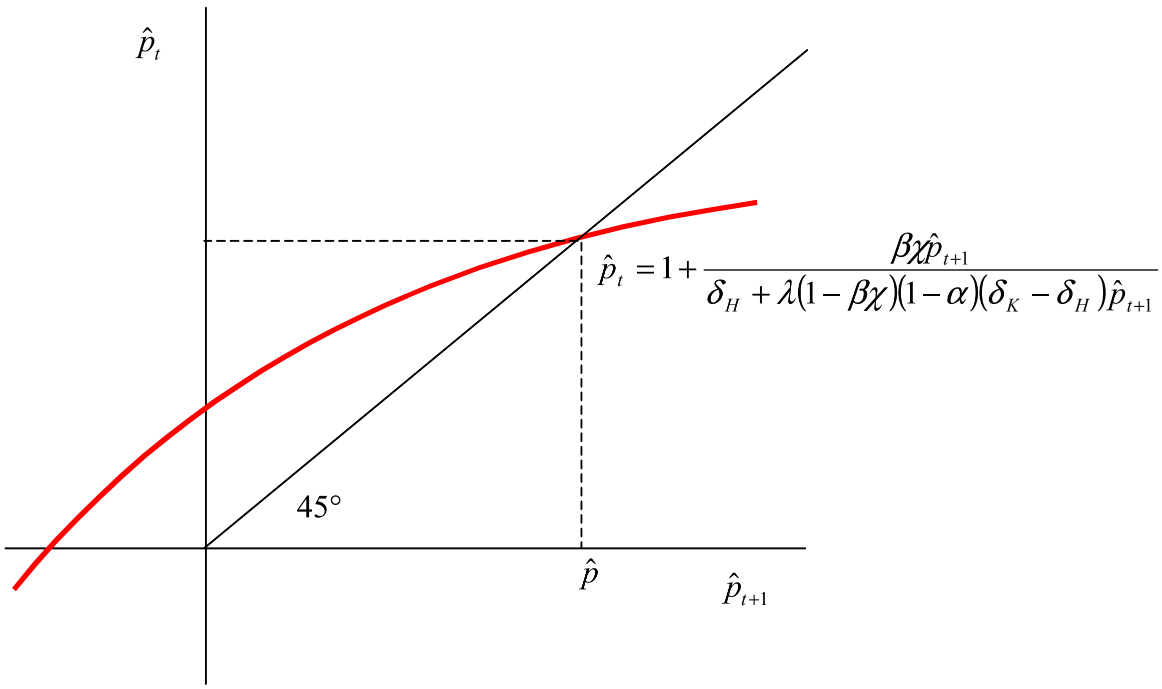

Figure 5

Figure 5 graphs the relationship between the price/earnings ratio in two adjacent periods for the case when there is no aggregate uncertainty. In the relevant parameter range the model implies that there is a unique determinate value for the price/earnings ratio denoted by $\hat{p}$. Determinacy means that there is a locally unique bounded sequence of values for p_t, given by the expression $\{\hat{p}_t = \hat{p}\}_{t=1}^{\infty}$ that satisfies the pricing equation. This implies that in this model, as in a representative agent economy, the ratio of the price of capital to the current realization of dividend uncertainty is a constant if agents have logarithmic preferences and if dividends follow a random walk.

But although the price of capital is fixed, it is not true that there is a unique way of pricing other assets. As I will show below, there is typically a multiplicity of ways of pricing assets as a consequence of the fact that the safe rate of return may be less than the growth rate. In a version of this economy with no uncertainty, Equation (22) can be used to construct a steady state relationship between the price/earnings ratio and the rate of return. This steady state relationship is given by the expression;

$$\frac{R}{\gamma} = \frac{\delta_K (\tilde{p} + 1)}{\tilde{p}}$$

where

$$R = \frac{1}{Q}$$

is the interest factor. In the Lucas asset pricing model [3] the set of agents is fixed and capital does not depreciate. In this case $\delta_K = 1$, and the interest rate is always greater than the growth rate. In the economy that I have constructed, capital does depreciate and, importantly, the set of agents changes over time. As a consequence, the interest rate may be less than the growth rate. This fact has an important consequence for asset pricing since it permits the existence of asset price bubbles that die out exponentially. The existence of equilibria of this kind will be important since it offers a possible explanation for the excess volatility puzzle pointed out by Robert Shiller.

8 A Two Asset Economy

When the government issues nominal bonds, equilibria are characterized by a pair of equations that describe how the value of bonds and equity evolve over time. The expression for the pricing kernel, written in terms of the price earnings ratio and the debt ratio is reproduced in Equation (23).

$$Q_{t+1}(\gamma')\gamma_{t+1} = \frac{\beta\pi(\gamma')\chi}{\delta_H + \lambda(1-\beta\chi)\left((\delta_K - \delta_H)(1-\alpha)\hat{p}_{t+1} + (1-\delta_H)z_{t+1}\right)}. \tag{23}$$

In an economy with taxes on dividend income at rate t, the price/earnings ratio is determined by the Equation (24)

$$\hat{p}_t = (1-t) + \sum_{\gamma'} \delta_K Q_{t+1}(\gamma')\gamma_{t+1}\hat{p}_{t+1}, \tag{24}$$

and the value of debt is determined by Equation (25);

$$z_t = -d + \sum_{\gamma'} Q_{t+1}\left(\gamma'\right)\gamma_{t+1} z_{t+1}. \tag{25}$$

Substituting Equation (23) into (24) and (25) leads to the following pair of functional equations that describe asset values in equilibrium.

$$\begin{aligned} \hat{p}_t &= (1-t) + E_t\left\{\frac{\beta\chi\hat{p}_{t+1}}{\delta_H + \lambda\left(1-\beta\chi\right)\left(\left(\delta_K-\delta_H\right)\left(1-\alpha\right)\hat{p}_{t+1} + \left(1-\delta_H\right)z_{t+1}\right)}\right\}, \\ z_t &= -d + E_t\left\{\frac{\beta\chi z_{t+1}}{\delta_H + \lambda\left(1-\beta\chi\right)\left(\left(\delta_K-\delta_H\right)\left(1-\alpha\right)\hat{p}_{t+1} + \left(1-\delta_H\right)z_{t+1}\right)}\right\}. \end{aligned}$$

In the following section I will calibrate these equations to the U.S. data and I will show that in a non-stochastic version of the model, this pair of equations has a unique economically meaningful steady state that is locally a saddle. There is locally a one dimensional manifold in $\hat{p}, z$ space that converges to the steady state. But, since both $\hat{p}$ and z are "jump variables" the economy could equally well begin from any point on this manifold. In the economy with uncertainty, the indeterminacy of equilibrium implies that there exist "sunspot fluctuations" that is, movements in equity prices and the market value of government debt that are unrelated to fundamentals. By choosing a monetary policy in which inflation is fixed, sunspot fluctuations that affect z are pushed into long bond prices rather than into the value of new debt issues. As a consequunce, equilibria will display slow movements in the book value of the debt/equity ratio, just as we see in the U.S. data.

9 Calibration

I will choose the parameters of the model to fit first moments of the U.S. data for the period from 1950 through 1980. The model has ten parameters, $t, d, r, \lambda, \alpha, \gamma, \beta, \chi, \delta_K$, and δ_H. The parameters t, d and r are policy parameters. I set the tax rate on dividends, t, equal to 0.44 and the ratio of the deficit to GDP, d, to -0.012. The figure for t is reported by Prescott and McGrattan [5] and the value for d is the historical mean of the ratio of the primary federal deficit to GDP computed from the NIPA accounts. For r, I choose $r = 0.05$. This corresponds to analyzing the case of a five percent nominal perpetuity.

The parameters λ , α and γ are relatively standard in the real business cycle literature. I set $\gamma = 1.015$ which was the average growth factor of per-capita GDP. I set α equal to 0.66 which is labor's share of GDP and $\lambda = 0.33$ which implies from Equation (7) that the representative household will spend approximately 25% of its time in market activities. This leaves the four parameters χ, β, d_H and δ_K. The model cannot distinguish between χ and β since they enter symmetrically. Hence, I choose $\beta = 1$, a value that ascribes time preference to the probability of death. Setting β equal to some other value, say 0.97, would simply lower the estimate of χ by an offsetting amount. I have left the parameter in the model to facilitate comparisons with representative agent economies.

This leaves the parameters χ, δ_H and δ_K. One possible route would be to interpret these parameters literally as the death probability, and the fractions of capital and human capital held by the generation of new adults and everybody else. This is not a fruitful route to take since, interpreted in this way, the parameters are so close to unity that the model behaves much like a representative agent model. I will take a different route. I will ask the question: what values must I choose for the parameters χ, δ_H and δ_K in order to explain the asset market data? Specifically, I will match 1) the safe rate of return 2) the historical average for the price equity ratio and 3) the historical equity premium.

Suppose first that there is no uncertainty. In this case the steady state of the asset pricing equations can be expressed by the following three equations

$$\hat{p} = 1 - t + Q\gamma\delta_K\hat{p}. \tag{26}$$

$$z = -d + Q\gamma z \tag{27}$$

$$Q\gamma = \frac{\beta\chi}{\left[\delta_H + \lambda\left(1-\beta\chi\right)\left[\left(1-\alpha\right)\left(\delta_K-\delta_H\right)\hat{p} + \left(1-\delta_H\right)z\right]\right]} \tag{28}$$

The following table summarizes the calibrated parameters that I have discussed so far.

Table 1

Parameter	*Value*	*Description*
γ	1.015	Productivity growth factor
α	0.66	Labor's share of income
λ	0.33	Utility weight on consumption
t	0.44	Dividend tax rate
d	-0.012	Average primary deficit to GDP ratio

Inserting these values in equations (26-28) leads to three equations in the variables p, z, Q and the parameters χ, δ_H,and δ_K .

$$\begin{aligned} p &= 0.56 + pQ\,(1.015\,\delta_K) && (29)\\ z &= 0.012\ + 1.015zQ && (30)\\ 1.015Q &= \frac{\chi}{[\delta_H + 0.33\,(1-\chi)\,[0.34\,(\delta_K - \delta_H)\,p + (1-\delta_H)\,z]]} && (31) \end{aligned}$$

The safe rate of return in the data was equal on average to 0.97. This implies the restriction

$$\frac{1}{Q\gamma} = 0.97$$

which determines the value of Q to be 1.0157. It follows from Equation (30) that $z = -0.3879$. The fact that the steady state market ratio of debt to GDP will be negative is implied by a policy in which the government retires debt each year at the rate that was historically observed in the period from 1950 through 1980.[4]

To calibrate $\hat{p}$ I choose a value of 14 which was the historical average for the period 1950-1979. Substituting the a value of value of $z = -0.39$ and $\hat{p} = 14$ into equations (29) and (31) leads to two equations in the three parameters χ, δ_K and δ_H.

$$\begin{aligned} 14 &= 0.56 + 14.433\delta_K && (32)\\ 1.0309 &= \frac{\chi}{[\delta_H + 0.33\,(1-\chi)\,[4.76\delta_K - 4.3721\delta_H - 0.3879]]} \end{aligned}$$

To determine the values of all three parameters I need one more observation. I will use the fact that historically, the return to equity has been approximately 5% higher than the inflation adjusted return to treasury bills. The following paragraph explains how the model can be parameterized to fit this fact.

In a risky environment, the premium on an asset is determined by its covariance with the pricing kernel. In simple real business cycle environments, the price/earnings ratio is a constant and all of the volatility in the pricing

[4] It is not important for the properties of the model that z should be negative in steady state equilibrium; rather it is an implied feature of calibrating the model to data in which the interest rate is less than the growth rate and the government is running, on average, a small primary surplus. In the data, debt has always been positive although, the debt to GDP ratio was converging to a negative value during the period from 1950 through 1980.

kernel arises as a consequence of fluctuations in consumption growth. The rate of return to capital is also much too smooth. In the next section I will exploit the fact that equilibrium are indeterminate to construct fluctuations in the price of equity that arise as a consequence of extrinsic uncertainty, (sunspots in the language of Cass and Shell [2]). By choosing the volatility of sunspot fluctuations to match observed movements in the price of equity the model will explain excess volatility in the sense of Shiller. However, to generate a large equity premium we need something more. It must also be the case that the pricing kernel, $Q_{t+1}\left(\gamma_{t+1}\right)$, is highly volatile.

The Sharpe ratio is defined as the ratio of the excess return of a risky asset to its standard deviation.[5] Using the notation R^R for the return to a risky asset, R^S for the risk-free rate, $\sigma\left(Q\right)$ for the standard deviation of the pricing kernel and $\sigma\left(R^R\right)$ for the standard deviation of the risky asset, the Sharpe ratio, is defined as,

$$\frac{E\left[R^R\right]-R^S}{\sigma\left[R^R\right]}.$$

Hansen and Jagannathan (HJ) pointed out that this ratio is bounded above by the standard deviation of the pricing kernel as a consequence of the inequality,

$$\frac{E\left[R^R\right]-R^S}{\sigma\left[R^R\right]}\leq\frac{\sigma\left[Q\right]}{E\left[Q\right]}. \tag{33}$$

The upper bound of this inequality is attained when the return to a risky asset has a correlation with Q of -1. Notice that the HJ bound implies that the equity premium on any risky asset can be no greater than the standard deviation of the pricing kernel, relative to its mean. In the data the Sharpe ratio has a historical average of 0.5 which implies that standard deviation of the pricing kernel must be equal to 50% of its mean.

In a representative agent model Q depends only on consumption growth and a volatile Q requires very high risk aversion of the representative agent. In my model, Q also depends on the price/earnings ratio $\hat{p}_t$ and on z_t, the market value of the debt to GDP ratio. Since sunspot fluctuations will be the main source of price volatility in the model, the correlation of the pricing

[5] See the discussion in Cochrane, [1] Chapter 1.

kernel with the return to equity will be close to -1. Hence, inequality (33) will be close to being attained as an equality. But to ensure a high value for the equity premium it is still necessary to choose parameters such that the standard deviation of Q is high. The pricing kernel in the model is related to $\hat{p}$ and z by the approximation (obtained from log linearizing Equation (28)).

$$\frac{dQ}{Q} = \frac{\lambda(1-\beta\chi)(1-\alpha)(\delta_K-\delta_H)\tilde{p}}{\delta_H+\lambda(1-\beta\chi)\left[(1-\alpha)(\delta_K-\delta_H)\hat{p}+(1-\delta_H)z\right]}\frac{d\hat{p}}{\hat{p}}.$$

Using this expression gives the third equation that I use to calibrate the model:

$$0.1 = \frac{\lambda(1-\beta\chi)(1-\alpha)(\delta_K-\delta_H)\tilde{p}}{\delta_H+\lambda(1-\beta\chi)\left[(1-\alpha)(\delta_K-\delta_H)\hat{p}+(1-\delta_H)z\right]}. \tag{34}$$

By picking the value of this expression to equal 0.1, a volatility of the price/earnings ratio of 5 will imply a volatility for Q of 0.5. Since the mean of Q is approximately 1, this is the correct order of magnitude necessary to generate a Sharpe ratio of 0.5.

Table 2 summarizes the moments of the data that I will use to pick parameters.

Table 2: Facts Used to Pick Parameters in Table 2

Parameter	*Value*	*Description*
R	0.97γ	Safe rate of interest
$\hat{p}$	14	Average price-earnings ratio
$\frac{E\left[R^R\right]-R^S}{\sigma(R^R)}$	0.5	Sharpe ratio

Equations (32-34) have two solutions,

$$\begin{aligned}\delta_K &= 0.931\,20, \quad \chi = 0.790\,49, \quad \delta_H = 0.698\,21\\ \delta_K &= 0.931\,20, \quad \chi = 1.338\,0, \quad \delta_H = 1.175\,7\end{aligned}$$

of which only one is relevant since χ and δ_H must be between zero and one. Table 3 summarizes the solutions values for parameters implied by calibrating these Equations (32-34) to fit the facts in Table 2.

Table 3: Calibrated Parameters

Parameter	*Value*	*Description*
χ	0.79	Survival probability
δ_K	0.93	$1-$ depreciation rate of physical capital
δ_H	0.69	$1-$depreciation rate of human capital

The value of $\chi = 0.93$ is consistent with a depreciation rate of 7% which is of the right order of magnitude for a depreciation rate for aggregate capital: But the values for χ and δ_H are too small to be taken literally. If χ is to be interpreted as a survival probability we would expect to see 20% of the population dying in given year which implies an expected lifetime of approximately 5 years. Similarly, a value for δ_H implies that human capital depreciates by 30% per year. Instead I propose a much more liberal interpretation of these parameters. Equity is priced only by active participants in the financial markets and the set of agents that participate in these markets has undergone substantial changes over the past three decades. These parameters may be more reasonable characterizations of the way that market participants change over time.

10 Simulated Data

As a first pass at how well the model performs I simulated data from the model, linearized around the steady state $\hat{p} = 14, \quad z = -0.39$. This is not an ideal solution since the model is highly non-linear and it is not clear that the approximation will be good over the range of observed fluctuations in rates of return. However, the method gives a first approximation to the features that the compete nonlinear model is likely to display. Equations (35) represent the linearized model

$$\begin{aligned} a_1 p_t &= Q_{t+1} + \gamma_{t+1} + p_{t+1} + w^1_{t+1} \\ a_2 z_t &= Q_{t+1} + \gamma_{t+1} + z_{t+1} + w^2_{t+1} \\ Q_{t+1} + \gamma_{t+1} &= a_3 p_{t+1} + a_4 z_{t+1} \end{aligned} \tag{35}$$

where the parameters a_1, a_2, a_3 and a_4 are given by the expressions:

$$\begin{aligned}
a_1 &= \frac{\hat{p}}{\hat{p}-(1-t)} \\
a_2 &= \frac{z}{z+d} \\
a_3 &= -\frac{\lambda(1-\beta\chi)(1-\alpha)(\delta_K-\delta_H)\hat{p}}{\delta_H+\lambda(1-\beta\chi)\left[(1-\alpha)(\delta_K-\delta_H)\hat{p}+(1-\delta_H)z\right]} \\
a_4 &= -\frac{\lambda(1-\beta\chi)(1-\delta_H)z}{\delta_H+\lambda(1-\beta\chi)\left[(1-\alpha)(\delta_K-\delta_H)\hat{p}+(1-\delta_H)z\right]},
\end{aligned}$$

and w_{t+1}^1 and w_{t+1}^2 are expectational errors defined by the expressions

$$\begin{aligned}
w_{t+1}^1 &= E\left\{Q_{t+1}+\gamma_{t+1}+p_{t+1}\right\}-Q_{t+1}+\gamma_{t+1}+p_{t+1} \\
w_{t+1}^2 &= E\left\{Q_{t+1}+\gamma_{t+1}+z_{t+1}\right\}-Q_{t+1}+\gamma_{t+1}+z_{t+1}
\end{aligned}$$

In matrix form

$$\begin{bmatrix} p_t \\ z_t \end{bmatrix} = A\begin{bmatrix} p_{t+1} \\ z_{t+1} \end{bmatrix} + \begin{bmatrix} \gamma_{t+1}+w_{t+1}^1 \\ \gamma_{t+1}+w_{t+1}^2 \end{bmatrix}$$

where

$$A = \begin{bmatrix} \frac{1+a_3}{a_1} & \frac{a_4}{a_1} \\ \frac{a_3}{a_2} & \frac{1+a_4}{a_2} \end{bmatrix}.$$

For the parameterization that I chose, A has a Jordan form

$$\begin{bmatrix} 1.0367 & -3.6740\times 10^{-2} \\ 0.62241 & -0.62241 \end{bmatrix}\begin{bmatrix} 0.87009 & 0 \\ 0 & 1.0357 \end{bmatrix}\begin{bmatrix} 1.0 & -5.9028\times 10^{-2} \\ 1.0 & -1.6657 \end{bmatrix}$$

with one eigenvalue, equal to 0.87009 inside the unit circle and the other, equal to 1.035 outside. The steady state is a saddle and any bounded solution must have the property that

$$1.0 \quad -5.9028\times 10^{-2} \begin{bmatrix} d\hat{p}_t \\ dz_t \end{bmatrix} = 0.$$

where $d\hat{p}_t$ and dz_t are proportional deviations for the steady state. However, there are many solutions of this kind, indexed by the probability distribution of the errors $\{w_t^1\}_{t=1}^{\infty}$ and $\{w_t^2\}_{t=1}^{\infty}$.[6] To simulate data for the price earnings ratio I set $z_t = \lambda_1 p_t$ where $\lambda_1 = (5.9028 \times 10^{-2})^{-1}$ sets $\{z_t, \hat{p}_t\}$ pairs on the stable manifold. Substituting this value for z_t back into Equation (35) gives the first order difference equation

$$p_{t+1} = \left(\frac{a_1}{1 + a_3 + a_4 \lambda_1} \right) p_t + w_{t+1}$$

To simulate this equation I chose $\{w_t\}$ to be a sequnce of normal i.i.d. random variables with standard deviation equal to 0.2.

To simulate data for the book value of the debt/GDP ratio, I linearized Equations (36) and (37).

$$\gamma_{t+1} b_{t+1} = \delta b_t \left(1 + \frac{r}{q_t}\right) + \frac{d}{q_t} \tag{36}$$

$$z_t = (q_t + r)\, b_t. \tag{37}$$

In any period one can treat b_t as predetermined and Equation (37) then determines the value of q_t . Given the realization of γ_{t+1}, Equation (36) can then be used to generate the subsequent value for b_{t+1}. The simulated data in Figure 6 was generated in this way.

[6] These errors are not independent since they must be related by the same restriction that ties z_t and $\hat{p}_t$.

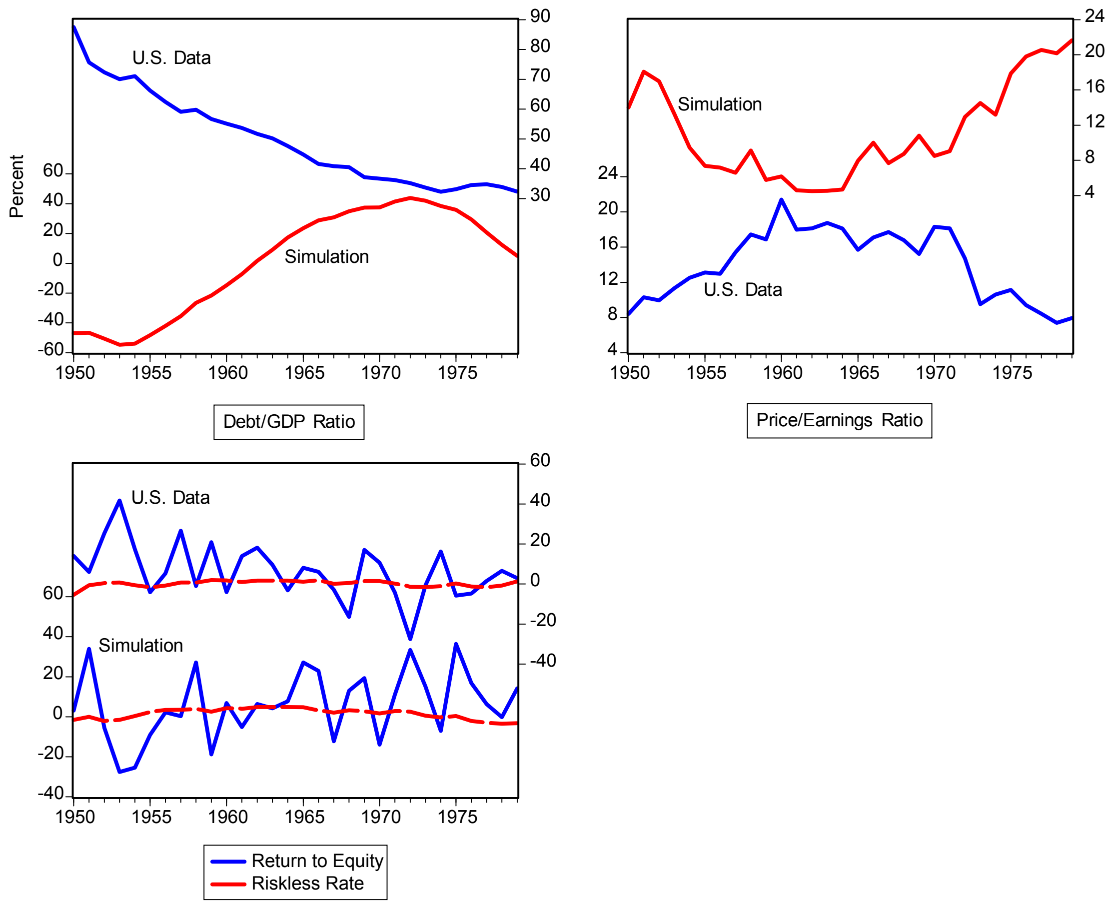

Figure 6

The top left panel plots a single draw of a simulation of the model for the debt/GDP ratio and compares it with the U.S. data. The top right panel compares a simulated draw of the price/earnings ratio with actual data and the bottom left panel plots the return to equity and the risk free rate. The data was simulated by drawing i.i.d. random normal variates for $\{\gamma_t\}$ with standard deviation of 0.02 to match the volatility of consumption growth in the data. For the sequence of sunspot shocks I added a separate i.i.d. sequence of random variables with standard deviation of 0.2, ten times more volatile than the consumption growth shocks. I chose $\sigma(w)$ to match the observed volatility of returns to equity in the data. Using these numbers, the covariance of the pricing kernel with the return to equity is approximately -0.05 implying that in the exact nonlinear model one would observe, on average, an equity premium of 5%.

There are several features of the simulated data that are important. First, the return to equity is extremely volatile and as a consequence the value of equity fluctuates far more than would be implied in a world in which the discount factor was fixed. Although the model was calibrated to have a mean equity premium of 0.5, the variance of returns is so high that the standard deviation of fifty year averages is of the order of 2.3. This is consistent with the data and it implies that long runs of returns well below the safe return are quite likely. Simulated data may display long runs in which the return to equity is consistently above the return to debt. But it may also display periods in which the return to equity is below the riskless rate for several periods in a row. Notice that in the simulated data the debt to GDP ratio displays local peaks like the one that we observed in 1990 in the U.S. data. This is a consequence of the fact that monetary policy feeds back onto fiscal policy through the effect of bond prices on the value of new debt issues. Also notice that the simulated price/dividend ratio is almost a random walk and that periods of long bull markets may be followed by long periods of market contractions.

11 Conclusion

The model that I have described is relatively simple; but it is able to capture many of the features of the U.S. data.

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