Chapter 2

1. The economic model we are considering consists of the equations

$$y_t = E_t\left[f\left(y_{t+1}, x_{t+1}, u_t\right)\right],$$

$$x_t = g\left(x_{t-1}, y_{t-1}\right),$$

$$x_0 = \bar{x}_0, \quad y_0 = \bar{y}_0.$$

To linearize this model we must first find a suitable point around which to take a Taylor series approximation. A good candidate for such a point is a stable steady state of the non-stochastic version of the model.

1.a We first assume that there exists a pair of values $\{\bar{x}, \bar{y}\}$ that solves the system of equations.

(1.a-1)
$$\begin{aligned}\bar{y} &= f\left(\bar{y}, \bar{x}, \bar{u}\right) \\ \bar{x} &= g\left(\bar{x}, \bar{y}\right)\end{aligned},$$

where $\bar{u}$ is the unconditional mean of u_t.

Now, define the parameters f_1, f_2, f_3, g_1 and g_2 as follows:

(1.a-2)
$$\begin{aligned}& f_1 = \left.\frac{\partial f}{\partial y}\right|_{\bar{y},\bar{x},\bar{u}}, \quad f_2 = \left.\frac{\partial f}{\partial x}\right|_{\bar{y},\bar{x},\bar{u}}, \quad f_3 = \left.\frac{\partial f}{\partial u}\right|_{\bar{y},\bar{x},\bar{u}} \\ & g_1 = \left.\frac{\partial g}{\partial y}\right|_{\bar{y},\bar{x},\bar{u}}, \quad g_2 = \left.\frac{\partial g}{\partial x}\right|_{\bar{y},\bar{x},\bar{u}}\end{aligned}$$

and write the linearized non-stochastic system as

(1.a-3)
$$A\begin{pmatrix} dy_t \\ dx_t \end{pmatrix} = B\begin{pmatrix} dy_{t+1} \\ dx_{t+1} \end{pmatrix} + C du_t,$$

where

(1.a-4)
$$A \equiv \begin{pmatrix} 1 & 0 \\ g_1 & g_2 \end{pmatrix}, \quad B \equiv \begin{pmatrix} f_1 & f_2 \\ 0 & 1 \end{pmatrix}, \quad C \equiv \begin{pmatrix} f_3 \\ 0 \end{pmatrix}.$$

For the linear approximation to be good this steady state should be stable. This implies that the roots of the matrix:

(1.a-5) $$J \equiv B^{-1}A,$$

should lie inside the unit circle.

Finally, in order for the stochastic model to "stay close to" the fixed point of the nonlinear model, we require that the random variable u_t should have a small bounded support.

1.b The point $\{\bar{x}, \bar{y}, \bar{u}\}$ as defined in part (a) is a good point around which to linearize the model because sequences that begin close to this point will remain close; for that purpose we needed a stable fixed point. The continued proximity to this point ensures that the approximation will remain constant as time progresses, i.e. as iteration of the dynamic system goes on.

1.c To derive a log linear model write the stochastic system in the form:

(1.c-1)
$$\begin{aligned} \ln(y_t) &= \ln\left[f\left(y_{t+1}, x_{t+1}, u_t\right)\right], \\ \ln(x_{t+1}) &= \ln\left[g(x_t, y_t)\right]. \end{aligned}$$

and define the parameters

(1.c-2)
$$\begin{aligned} &\hat{f}_1 = \frac{\bar{y} f_1}{f\left(\bar{y}, \bar{x}, \bar{u}\right)}, \quad \hat{f}_2 = \frac{\bar{x} f_2}{f\left(\bar{y}, \bar{x}, \bar{u}\right)}, \quad \hat{f}_3 = \frac{\bar{u} f_3}{f\left(\bar{y}, \bar{x}, \bar{u}\right)}, \\ &\hat{g}_1 = \frac{\bar{y} g_1}{f\left(\bar{y}, \bar{x}, \bar{u}\right)}, \quad \hat{g}_2 = \frac{\bar{x} g_2}{g\left(\bar{y}, \bar{x}, \bar{u}\right)}, \end{aligned}$$

where f_1, f_2, f_3, g_1 and g_2 are as defined in part (a).

Then the log linear approximation is given by:

(1.c-3)
$$\begin{aligned} \ln(y_t) &= E_t\left[\hat{f}_1 \ln(y_{t+1}) + \hat{f}_2 \ln(x_{t+1}) + \hat{f}_3 \ln(u_t)\right] + k_1, \\ \ln(x_t) &= \hat{g}_2 \ln(x_{t-1}) + \hat{g}_1 \ln(y_{t-1}) + k_2, \end{aligned}$$

where k_1 and k_2 are constants.

1.d Since the model delivers two initial conditions, $\bar{x}_0$and $\bar{y}_0$ any steady state (as long as one exists) is *always* determinate. Solutions that begin close to a steady state may diverge away from it and remain bounded. If local solutions diverge from the steady state then the linear approximation will be inaccurate but local solutions to the dynamic equations could *still* be determinate equilibria. To find if any particular solution is a determinate equilibrium we would need to know more about the non-linear map in order to know whether a particular isolated trajectory remains bounded.

2. Since elasticities are logarithmic derivatives, it helps to express the functions to be linearized in the following way:

a) $\ln(y_t) = \dfrac{1}{\theta}\ln\left(a x_t^{\theta} + (1-a) z_t^{\theta}\right)$

b) $\ln(y_t) = \dfrac{\gamma}{\theta}\ln\left(a x_t^{\theta} + (1-a) z_t^{\theta}\right)$

c) $\ln(y_t) = \ln(a x_t + b z_t)$

d) $\ln(y_t) = \alpha \ln(x_t) + \beta \ln(z_t)$

e) $\ln(y_t) = \ln\left(z_t^{\alpha} x_t^{\beta} + 1\right)$

The elasticities of these functions are logarithmic partial derivatives:

a) $\varepsilon_{fx} = a\left(\dfrac{\bar{x}}{\bar{y}}\right)^{\theta}$, $\qquad \varepsilon_{fz} = (1-a)\left(\dfrac{\bar{z}}{\bar{y}}\right)^{\theta}$,

b) $\varepsilon_{fx} = a\gamma\left(\dfrac{\bar{x}}{\bar{y}^{1/\gamma}}\right)^{\theta}$, $\qquad \varepsilon_{fz} = (1-a)\gamma\left(\dfrac{\bar{z}}{\bar{y}^{1/\gamma}}\right)^{\theta}$,

c) $\varepsilon_{fx} = a\left(\dfrac{\bar{x}}{\bar{y}}\right)$, $\qquad \varepsilon_{fz} = b\left(\dfrac{\bar{z}}{\bar{y}}\right)$,

d) $\varepsilon_{fx} = \alpha$, $\qquad \varepsilon_{fz} = \beta$,

e) $\varepsilon_{fx} = \beta\left(\dfrac{\bar{y}-1}{\bar{y}}\right)$ $\qquad \varepsilon_{fz} = \alpha\left(\dfrac{\bar{y}-1}{\bar{y}}\right)$.

3. Consider the matrix,

$$A = \begin{bmatrix} 4 & \sqrt{3} \\ \sqrt{3} & 2 \end{bmatrix}.$$

3.a The eigenvalues of *A* are obtained by solving the characteristic equation:

(3.a-1) $$p(\lambda) = |A - \lambda I| = \begin{vmatrix} 4-\lambda & \sqrt{3} \\ \sqrt{3} & 2-\lambda \end{vmatrix} = 0$$

$\Rightarrow \lambda^2 - 6\lambda + 5 = 0$

thus $\lambda = 1$ *or* $\lambda = 5$.

To obtain eigenvector x^1 of A (associated with the eigenvalue $\lambda^1 = 1$), one solves the equations: NB (superscripts index eigenvalues and eigenvectors):

(3.a-2) $$Ax^1 = \lambda^1 x^1.$$

(3.a-3) $$\begin{bmatrix} 4 & \sqrt{3} \\ \sqrt{3} & 2 \end{bmatrix} \begin{bmatrix} x_1^1 \\ x_2^1 \end{bmatrix} = \begin{bmatrix} x_1^1 \\ x_2^1 \end{bmatrix}$$

(3.a-4) $$\begin{cases} 4x_1^1 + \sqrt{3}x_2^1 = x_1^1 \\ \sqrt{3}x_1^1 + 2x_2^1 = x_2^1 \end{cases}$$

Since this system is singular we can choose either of the equations to solve for the elements of x^1. Selecting the second row gives x_1^1 in terms of x_2^1:

(3.a-5) $$\sqrt{3}x_1^1 = -x_2^1.$$

Similarly, to solve for the eigenvector x^2 associated with eigenvalue $\lambda^2 = 5$, one solves the equations:

(3.a-6) $$Ax^2 = \lambda^2 x^2$$

(3.a-7) $$\begin{bmatrix} 4 & \sqrt{3} \\ \sqrt{3} & 2 \end{bmatrix} \begin{bmatrix} x_1^2 \\ x_2^2 \end{bmatrix} = 5 \begin{bmatrix} x_1^2 \\ x_2^2 \end{bmatrix}$$

(3.a-8)
$$\begin{cases} 4x_1^2 + \sqrt{3}x_2^2 = 5x_1^2 \\ \\ \sqrt{3}x_1^2 + 2x_2^2 = 5x_2^2 \end{cases}$$

Selecting the first equation to express x_2^2 in terms of x_1^2 we obtain:

(3.a-9)
$$\sqrt{3}x_2^2 = x_1^2.$$

The requirement that $Q^{-1} = Q'$ implies that

1) $Det(Q) = -1$
2) $q_{11} = -q_{22} \quad \Rightarrow \quad x_1^1 = -x_2^2,$
3) $q_{12} = q_{21} \quad \Rightarrow \quad x_2^1 = x_1^2.$

Using these facts we can write Q as:

(3.a-10)
$$Q' = \begin{bmatrix} y & z \\ z & -y \end{bmatrix} = Q^{-1}.$$

Given a determinant of -1 together with (3.a-9) gives

(3.a-11)
$$\begin{aligned} &y^2 + z^2 = 1, \quad -\sqrt{3}y = z, \\ &\Rightarrow y = \pm 1/2, \quad z = \mp\sqrt{3}/2. \end{aligned}$$

Selecting the values $y = -1/2$ and $z = \sqrt{3}/2$ gives matrices Q and Λ which diagonalize A:

(3.a-12)
$$Q = \begin{bmatrix} -1/2 & \sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{bmatrix}, \quad \Lambda = \begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix}.$$

3.b Using the same procedure as above, the matrices Q and Λ that diagonalize

$$A = \begin{bmatrix} 3 & \sqrt{2} \\ \sqrt{2} & 2 \end{bmatrix}$$ are

(3.b-1)
$$Q = \begin{bmatrix} -1/\sqrt{3} & \sqrt{2}/\sqrt{3} \\ \sqrt{2}/\sqrt{3} & 1/\sqrt{3} \end{bmatrix}, \quad \Lambda = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}.$$

4. To solve this problem, first diagonalize A.

4.a The characteristic polynomial of A is:

(4.a-1)
$$\begin{aligned}&p(\lambda)=(1/2-\lambda)(1/2-\lambda)-1/16=0\\&\Rightarrow \lambda^2-\lambda+3/16=0\\&\Rightarrow \lambda_1=1/4,\quad \lambda_2=3/4.\end{aligned}$$

Its eigenvectors solve:

(4.a-2)
$$\begin{bmatrix}1/2 & 1\\ 1/16 & 1/2\end{bmatrix}\begin{bmatrix}1\\ x\end{bmatrix}=1/4\begin{bmatrix}1\\ x\end{bmatrix} \text{ and}$$
$$\Rightarrow x=-1/4,$$

(4.a-3)
$$\begin{bmatrix}1/2 & 1\\ 1/16 & 1/2\end{bmatrix}\begin{bmatrix}1\\ y\end{bmatrix}=3/4\begin{bmatrix}1\\ y\end{bmatrix}$$
$$\Rightarrow y=1/4,$$

where we have normalized the first element of each eigenvector to equal 1. The matrix Q of stacked eigenvectors is then:

(4.a-4)
$$Q=\begin{bmatrix}1 & 1\\ -1/4 & 1/4\end{bmatrix}$$

and its inverse is

(4.a-5)
$$Q^{-1}=\begin{bmatrix}1/2 & -2\\ 1/2 & 2\end{bmatrix}$$

and the diagonalization of A is represented as

(4.a-6)
$$A=\begin{bmatrix}1 & 1\\ -1/4 & 1/4\end{bmatrix}\begin{bmatrix}1/4 & 0\\ 0 & 3/4\end{bmatrix}\begin{bmatrix}1/2 & -2\\ 1/2 & 2\end{bmatrix}.$$

Using this result we can write the system as

(4.a-7)
$$\begin{aligned}z_t^1&=1/4z_{t-1}^1+w_t^1, && z_0^1=0,\\ z_t^2&=3/4z_{t-1}^2+w_t^2, && z_0^2=0.\end{aligned}$$

where

(4.a-8) $$\begin{bmatrix} z_t^1 \\ z_t^2 \end{bmatrix} = \begin{bmatrix} 1/2 & -2 \\ 1/2 & 2 \end{bmatrix} \begin{bmatrix} y_t \\ x_t \end{bmatrix}, \quad \begin{bmatrix} w_t^1 \\ w_t^2 \end{bmatrix} \equiv \begin{bmatrix} 1/2 & -2 \\ 1/2 & 2 \end{bmatrix} \begin{bmatrix} u_t \\ v_t \end{bmatrix}.$$

4.b To find the support of x and y first note that the supports of w^1 and w^2 are

(4.b-1) $$w_t^1 \in [-5/2, 5/2], \quad w_t^2 \in [-5/2, 5/2].$$

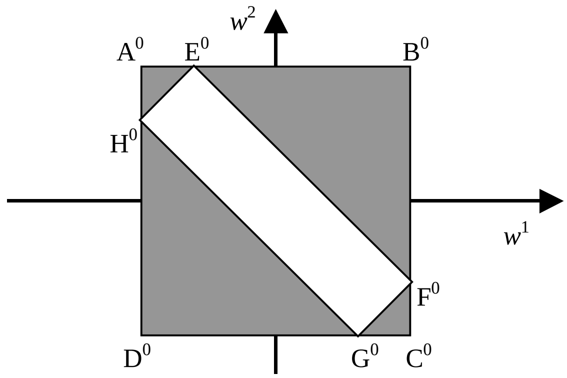


Figure 2.1: The invariant support of w^1 and w^2

Figure 2.1 shows the support of w^1 and w^2. The shaded area is the minimal product space which covers the joint support, shown as the white rectangle. The corners of the joint support are convex combinations of neighboring extreme points of the minimal product space. The weights in these convex combinations will play an important role in what follows and are 1/5 and 4/5 in our case. For instance, $E^0 = 4/5A^0 + 1/5B^0$.

The supports of the invariant distributions of z^1 and z^2 are easily found by solving equations of the following form for z_b^i.

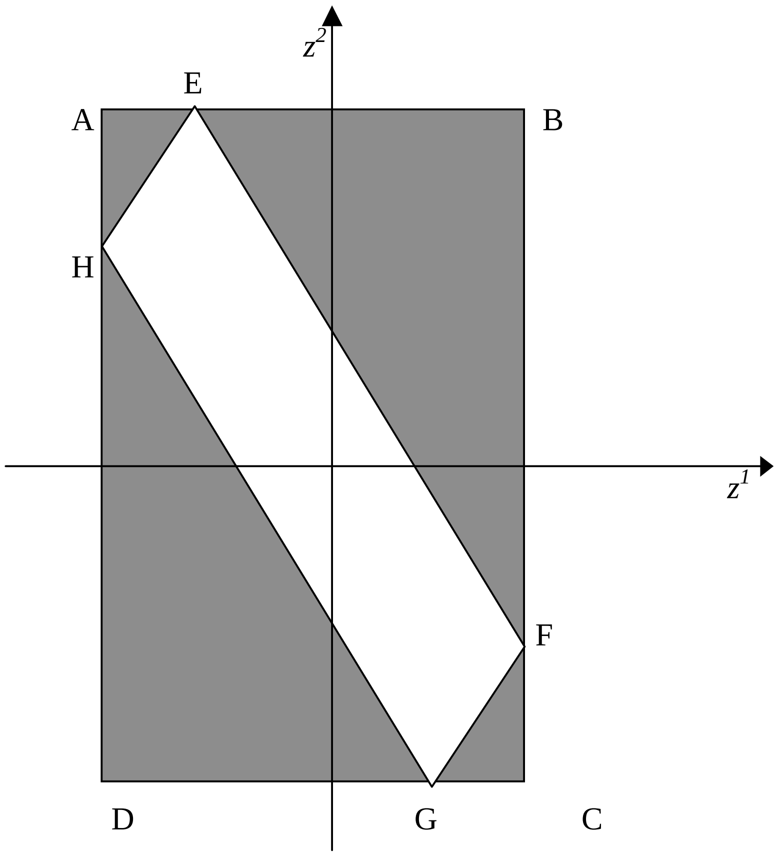
(4.b-2) $$z_b^i = \lambda^i z_b^i + w_b^i, \qquad i = 1, 2;\ b = l, u \text{ (i.e. lower, upper)}$$

where λ^i are the eigenvalues and w_b^i denotes upper and lower bounds of the errors in the diagonalized system. This is just the approach explained in the textbook on p. 20. We find:

(4.b-3) $$z_l^1 = -10/3,\ z_u^1 = 10/3,\ z_l^2 = -10,\ z_u^2 = 10.$$

This defines the minimal product space that covers the joint support.

Figure 2.2 shows the support of z^1, z^2. The joint support is the white parallelogram, whose corners are obtained as convex combinations of the extreme points A, B, C, D, again using the weights 1/5 and 4/5.



<u>Figure 2.2</u>: The invariant support of z^1 and z^2

Equation (4.a-8) tells us how we can map (back) from z^1 and z^2 into y and x.

(4.b-4)
$$\begin{bmatrix} y \\ x \end{bmatrix} = Qz = \begin{bmatrix} 1 & 1 \\ -1/4 & 1/4 \end{bmatrix} \begin{bmatrix} z^1 \\ z^2 \end{bmatrix}$$

In order to obtain the support of the invariant distribution of y and x (4.b-4) is applied to all points of the support of the invariant distribution of z^1 and z^2.

However, since Q is a linear mapping, it is sufficient to apply it to the corners (A, B, C, D) of the minimal product space that covers the invariant distribution of z^1 and z^2. That is, the new set (the support of the invariant distribution of y and x) can have a different shape but its boundary will still consist of a union of straight (not bent) lines. Points on these lines are convex combinations of P' = QP and R' = QR, where P and R are neighboring corners of the old support.

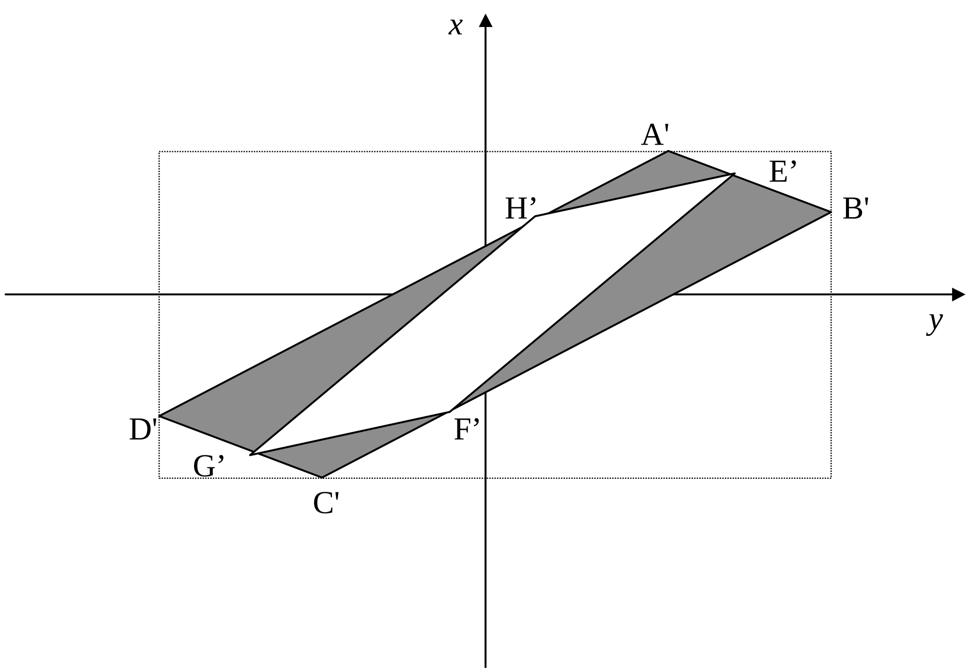
(4.b-5) $$A' = \begin{bmatrix} 1 & 1 \\ -1/4 & 1/4 \end{bmatrix} \begin{bmatrix} -10/3 \\ 10 \end{bmatrix} = \begin{bmatrix} 20/3 \\ 10/3 \end{bmatrix}$$

(4.b-6) $$B' = \begin{bmatrix} 1 & 1 \\ -1/4 & 1/4 \end{bmatrix} \begin{bmatrix} 10/3 \\ 10 \end{bmatrix} = \begin{bmatrix} 40/3 \\ 5/3 \end{bmatrix}$$

(4.b-7) $$C' = \begin{bmatrix} 1 & 1 \\ -1/4 & 1/4 \end{bmatrix} \begin{bmatrix} 10/3 \\ -10 \end{bmatrix} = \begin{bmatrix} -20/3 \\ -10/3 \end{bmatrix}$$

(4.b-8) $$D' = \begin{bmatrix} 1 & 1 \\ -1/4 & 1/4 \end{bmatrix} \begin{bmatrix} -10/3 \\ -10 \end{bmatrix} = \begin{bmatrix} -40/3 \\ -5/3 \end{bmatrix}$$

The joint support is the white parallelogram, whose corners are obtained as convex combinations of the extreme points A', B', C', D', again using the weights 1/5 and 4/5. We get: $E' = (8,3)$; $F' = (-2.\dot{6},-2.\dot{3})$; $G' = (-8,-3)$; $H' = (2.\dot{6},2.\dot{3})$.



<u>Figure 2.3</u>: The support of the invariant distribution of *y* and *x*

Figure 2.3 shows the joint support of the invariant distribution of *y* and *x* as the white parallelogram inside the gray one.

A more straightforward[1], however less instructive (as judged by the applications we are interested in in the next chapters), way of calculating the support starts with the linear economic model

(4.b-9) $$\begin{pmatrix} y_t \\ x_t \end{pmatrix} = A\begin{pmatrix} y_{t-1} \\ x_{t-1} \end{pmatrix} + \begin{pmatrix} u_t \\ v_t \end{pmatrix},$$

drops the time subscripts and solves for

(4.b-10) $$\begin{pmatrix} y \\ x \end{pmatrix} = (I - A)^{-1}\begin{pmatrix} u \\ v \end{pmatrix},$$

which can be used to map the extreme points of the support of the error terms into the support of y and x, to yield the same result as above for E', F', G', H'.

5. Equation (2.38) in the text can be rearranged to express future expectations in terms of current observations of x and y:

$$y_{t+1}^e = \frac{1}{\alpha} y_t - \frac{\beta}{\alpha} x_t - \frac{1}{\alpha} u_t .$$

5.a We can substitute out the unobservable expectations, y_{t+1}^e and y_t^e, from equation (2.40) in the text, to give a difference equation in terms of observable variables y_t and x_t and the error terms u_t and u_{t-1}:

(5.a-1) $$\begin{aligned} & y_{t+1}^e = \lambda y_t^e + (1-\lambda) y_t \\ & \Rightarrow \frac{1}{\alpha} y_t - \frac{\beta}{\alpha} x_t - \frac{1}{\alpha} u_t = \frac{\lambda}{\alpha} y_{t-1} - \frac{\lambda\beta}{\alpha} x_{t-1} - \frac{\lambda}{\alpha} u_{t-1} + (1-\lambda) y_t \\ & \Rightarrow y_t = \theta\left[\lambda y_{t-1} + \beta x_t - \beta\lambda x_{t-1} + u_t - \lambda u_{t-1}\right], \end{aligned}$$

where the parameter θ is defined as follows:

(5.a-2) $$\theta = \frac{1}{1 - \alpha(1-\lambda)} .$$

5.b The support of the distribution of x can be obtained from its law of motion,

(5.b-1) $$x_t = \gamma x_{t-1} + \delta + v_t .$$

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[1] This and an error in a previous version of the solutions manual, which has led to the "white parallelograms inside the gray ones", was pointed out to us by Stefan Weber, graduate student at the European University Institute (EUI), Florence, Italy.

Iterating this equation backwards, (note that $\gamma < 1$), we can express x_t as a function of past values of v_t's and the initial value, x_0:

(5.b-2) $$x_t = \gamma^t x_0 + \sum_{j=1}^{t} \delta\gamma^{j-1} + \sum_{j=1}^{t} \gamma^{j-1} v_{t-j+1}.$$

The lower and upper bounds of the support of x are calculated by taking the limits of (5.b-2) assuming that v takes on its lowest and highest values in every period. Thus, (applying the rule for the summation of a geometric series), we can show that $x \in [\underline{x}, \overline{x}]$, where:

(5.b-3) $$\underline{x} = \frac{\delta - a}{1-\gamma}, \quad \overline{x} = \frac{\delta + a}{1-\gamma}.$$