Chapter 5

1. The present value price of a period *s* commodity in period *t* is its relative price in terms of period *t* commodities. Thus if R_t is the nominal interest factor on period *t* bonds, the present value price of a period *5* commodity is

$$Q_1^5 = \frac{p_5}{\prod_{t=1}^{4} R_t}.$$

2. The representative agent maximizes lifetime utility

$$= \sum_{t=1}^{\infty} \beta^{t-1}\left[C_t - L_t^{\gamma}\right], \quad 0 \le \beta < 1, \quad \gamma > 1,$$

subject to

$$K_{t+1} = (1-\delta)K_t + Y_t - C_t,$$

$$Y_t = K_t^{\alpha} L_t^{1-\alpha}.$$

2.a The objective function for this problem is

$$\mathcal{L} = \sum_{t=1}^{\infty} \beta^{t-1}\left[K_t^{\alpha} L_t^{1-\alpha} + (1-\delta)K_t - K_{t+1} - L_t^{\gamma}\right], \tag{2.a-1}$$

obtained by substituting the production function into the period resource constraint and then using the latter to substitute for consumption in the utility function. Notice that utility is linear in consumption which implies that the consumer is indifferent as to when he consumes (there is no consumption smoothing motive). The first order conditions for the choice of L_t and K_{t+1} are:

$$L_t: \quad \gamma L_t^{\gamma} = (1-\alpha)Y_t, \tag{2.a-2}$$

$$K_{t+1}: \quad 1 = \alpha\beta \frac{Y_{t+1}}{K_{t+1}} + \beta(1-\delta). \tag{2.a-3}$$

The four equations that characterize the competitive equilibrium are (2.a-2), (2.a-3), the period resource constraint (2.a-4)

$$C_t = Y_t + (1-\delta)K_t - K_{t+1}, \tag{2.a-4}$$

and the production function (2.a-5)

(2.a-5) $$Y_t = K_t^{\alpha} L_t^{1-\alpha}.$$

2.b The two first order conditions (2.a-2) and (2.a-3) are not sufficient to guarantee the optimality of the representative agent's consumption path. In finite time, we require that the representative agent's budget constraint should hold with equality, i.e., he should not die holding positive assets. The transversality condition is the infinite horizon version of this, and requires that the limit of the discounted value of representative agent's wealth in the infinite future, weighted by the marginal utility of consumption, be zero. For this problem, the transversality condition is

(2.b-1)
$$\lim_{T\to\infty}\left\{\beta^{T-1}\frac{\partial U}{\partial C_T}K_{T+1}\right\} = 0$$
$$\Leftrightarrow \lim_{T\to\infty}\left\{\beta^{T-1}K_{T+1}\right\} = 0.$$

2.c From (2.a-3), in the steady state,

(2.c-1) $$\bar{Y} = \left(\frac{1-\beta(1-\delta)}{\alpha\beta}\right)\bar{K}.$$

From (2.a-2) and (2.c-1)

(2.c-2) $$\bar{L} = \left[\frac{(1-\alpha)(1-\beta(1-\delta))}{\alpha\beta\gamma}\bar{K}\right]^{1/\gamma}.$$

Using (2.c-1), (2.c-2) and the production function (2.a-5),

(2.c-3)
$$\bar{Y} = \bar{K}^{\alpha}\bar{L}^{1-\alpha}$$
$$\Rightarrow \frac{1-\beta(1-\delta)}{\alpha\beta}\bar{K} = \bar{K}^{\alpha}\left(\frac{(1-\alpha)(1-\beta(1-\delta))}{\alpha\beta\gamma}\bar{K}\right)^{(1-\alpha)/\gamma}$$

rearranging terms gives:

(2.c-4)
$$\bar{K} = \left[\frac{\alpha\beta}{1-\beta(1-\delta)}\left(\frac{(1-\alpha)(1-\beta(1-\delta))}{\alpha\beta\gamma}\right)^{(1-\alpha)/\gamma}\right]^{\gamma/(\gamma-1)(1-\alpha)}$$

2.d To answer this question we must find a difference equation that establishes how the economy evolves through time. From equations (2.a-2) and (2.a-5) we can express labor as a function of K:

(2.d-1) $$\gamma L_t^{\gamma} = (1-\alpha) K_t^{\alpha} L_t^{1-\alpha} \;\Rightarrow\; L_t = \left(\frac{(1-\alpha)K_t^{\alpha}}{\gamma}\right)^{\frac{1}{\gamma+\alpha-1}}.$$

Substituting this expression back into the Euler equation (2.a-3) delivers the expression:

(2.d-2) $$1 = a_1 K_{t+1}^{(1-\alpha)\left(\frac{1-\gamma}{\gamma+\alpha-1}\right)} + \beta(1-\delta)$$

where a_1 is a function of the parameters of the model. Equation (2.d-2) reveals that the dynamics of capital accumulation in this model are trivial – adjustment to the steady state is instantaneous. The intuition for this result is that the consumer is indifferent as to when he consumes due to the assumption that utility is linear in consumption. Hence the economy jumps immediately to the optimal capital stock. Provided that the economy starts close enough to $\overline{K}$, labor supply will always equal its optimal level, consumption will jump to its optimal level after one period, taking up the adjustment slack in the initial period.

2.e The rental rate is $\rho_t = \alpha\left(\frac{K_t}{L_t}\right)^{\alpha-1}$, the interest rate is $r_t = \rho_t - \delta$, and the real wage rate $\omega_t = (1-\alpha)(K_t / L_t)^{\alpha}$. It follows that the time path of each of these variables can be calculated once we know how the capital labor ratio evolves over time. From (2.d-1) it follows that the capital labor ratio is given by the following function of the capital stock (where b is a compound parameter):

(2.e-1) $$\frac{K_t}{L_t} = bK_t^{\frac{\gamma-1}{\gamma+\alpha-1}}$$

This gives us a positive relationship between the capital labor ratio and K_t, since we assumed $\gamma > 1$ for concavity.

2.f If $\gamma = 1$, (2.d-2) becomes degenerate in K_{t+1}. This equation can no longer hold with equality (unless the marginal product of capital just happens to equal the rate of time preference). Generally, either the marginal product of capital will exceed the rate of time preference, or it will fall short. In the first case, the consumer will devote all resources to capital accumulation and he will never consume. In the second case, he will consume the entire capital stock in the initial period and the economy will shut down.

3. In the infinite horizon case we assume that preferences are additively separable, and that the period utility functions are strictly concave. In the finite case we assume quasi-convavity and do not impose separability.

3.a If we were to drop the assumption of a common discount rate, the equilibrium wealth distribution would be degenerate. The most patient individual would asymptotically own all of the wealth.

3.b In a two-country model, the more patient country (call this country A) would lend to the less patient one (country B). Initially, commodities would flow from A to B. Eventually, the balance of trade would be reversed as the debts of country B built up and the interest payments on the debt increased. Asymptotically, all of the output produced in country B will flow to country A as interest on the debt.