# Chapter 6

**1.** This question deals with the standard two period overlapping generations model with money.

**1.a** The period $t$ budget constraint for a young person born at date $t$, is $c_t^t + s_t \le a$, where $s_t$ is her saving. The period budget constraint for the same person at date $t$+1 is $c_{t+1}^t \le b + R_t s_t$. Combining these constraints, we obtain the life cycle budget constraint of a young person at date $t$

(1.a-1) $$c_t^t + \frac{c_{t+1}^t}{R_t} \le a + \frac{b}{R_t}.$$

**1.b** A young person at date $t$ maximizes her utility function $U_t = \log c_t^t + \beta \log c_{t+1}^t$ subject to (1.a-1), i.e., she maximizes

(1.b-1) $$\log(a - s_t) + \beta \log(b + R_t s_t),$$

by choosing her saving $s_t$. The first order condition for (1.b-1) when simplified using the budget constraint gives the following saving function:

(1.b-2) $$s(R_t) = \frac{\beta}{1+\beta} a - \frac{1}{1+\beta}\left(\frac{b}{R_t}\right).$$

**1.c** In equilibrium, the excess demands of the young and the old must sum to zero. The excess demand of the young in period $t$ is $x_t^t - a = -s(R_t)$, and the excess demand of the old in period $t$ is $x_t^{t-1} - b = R_{t-1} s(R_{t-1})$. Thus equilibrium sequences of $\{R_t\}$ satisfy the following difference equation:

(1.c-1) $$s(R_t) - R_{t-1} s(R_{t-1}) = 0 \Rightarrow R_t = \frac{b}{b + a\beta(1 - R_{t-1})}.$$

The initial condition is that in the first period, $t$ = 1, young agents hold their savings in money balances, $s(R_1) = M / p_1$.

**1.d** The steady-state interest factor $R$ satisfies the quadratic equation (1.d-1) below, derived from equation (1.c-1):

(1.d-1) $$R^2 - \left(\frac{b}{a\beta} + 1\right) R + \frac{b}{a\beta} = 0.$$

Thus the steady state interest factors are $\bar{R}_1 = b/a\beta, \bar{R}_2 = 1$. Of these, $\bar{R}_1$ is the autarkic interest factor, and $\bar{R}_2$ the golden rule. Money has value only at the golden rule. But for that we require that the economy is Samuelson, i.e.,

(1.d-2) $$s(\bar{R}_2) > 0 \Leftrightarrow a > \frac{b}{\beta}.$$

**1.e** For equilibrium to be unique, the economy must be Classical

(1.e-1) $$a \le \frac{b}{\beta}.$$

Yes the equilibrium is determinate. Consider Figure 6.1:

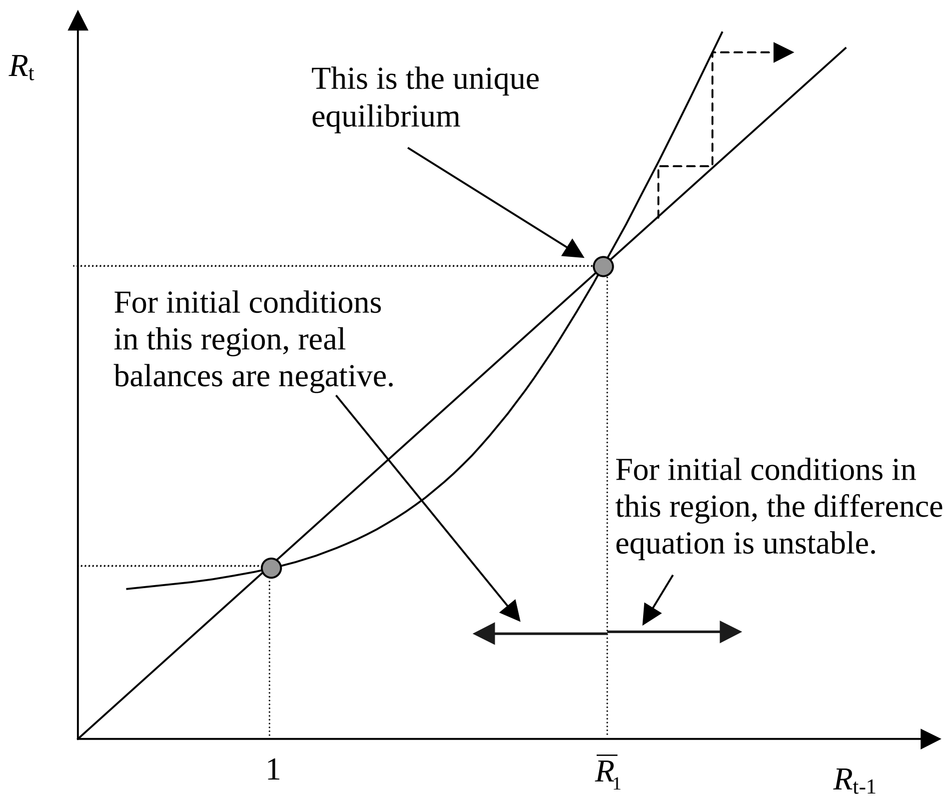


Figure 6.1

This figure illustrates that when the economy is Classical, there is a unique determinate equilibrium. Further, it is stationary. Dynamic equilibria that begin with an interest factor above the autarkic interest factor $\bar{R}_1$ are explosive. Dynamic equilibria that begin with a lower interest factor than $\bar{R}_1$ are infeasible since they would require negative real balances.

**2.** This question deals with an overlapping generations model with multiple types of agent.

**2.a** The intertemporal budget constraint of a type 1 agent is

(2.a-1) $$x_t^{1t} + \frac{x_{t+1}^{1t}}{R_t} = 1 \quad ,$$

and for a type 2 agent, it is

(2.a-2) $$x_t^{2t} + \frac{x_{t+1}^{2t}}{R_t} = \frac{1}{R_t} \, .$$

**2.b** Maximization of a type 1 agent's utility $U^1 = \log x_t^{1t} + \beta \log x_{t+1}^{1t}$ subject to (2.a-1) gives the following demand $\hat{x}_t^{1t} = 1/(1+\beta)$. Therefore, the excess demand of a type 1 agent is

(2.b-1) $$f^1(R_t) = -\frac{\beta}{1+\beta}.$$

Similarly, maximization of a type 2 agent's utility subject to (2.a-2) gives the following excess demand function:

(2.b-2) $$f^2(R_t) = \frac{1}{(1+\beta)R_t}.$$

Thus the aggregate excess demand of the young in period *t* is

(2.b-3) $$f(R_t) = \frac{1}{1+\beta}\left[\frac{1}{R_t} - \beta\right].$$

**2.c** A competitive equilibrium is a sequence of allocations for each type *j* $\left\{\left(x_t^{jt}, x_{t+1}^{jt}\right)\right\}_{t=1}^{\infty}$ and $\left\{\left(x_1^{j0}\right)\right\}$, *j* = 1, 2, and prices $\left\{(R_t)\right\}_{t=1}^{\infty}$, such that

**a)** $f(R_t) - R_{t-1} f(R_{t-1}) = 0, \quad \forall t \geq 1$,

**b)** $f(R_t) = \frac{1}{1+\beta}\left[\frac{1}{R_t} - \beta\right], \quad \forall t \geq 1$, and

**c)** $f(R_1) = 0.$

One possible equilibrium is characterized by the sequence of interest factors: $R_t = 1/\beta \quad \forall t$.

**2.d** The stationary equilibrium $R_t = 1/\beta \quad \forall t$ is the only equilibrium in this economy. Net savings of young agents is zero in this equilibrium.

**2.e** The unique equilibrium is efficient iff $\beta \le 1$.

**2.f** In the unique equilibrium, $R_t = 1/\beta$ members of the same generation trade amongst themselves as the two types of agents have different endowment profiles. There is, however, no trade between generations.

**2.g** Define $\pi_{t+1} = p_{t+1} / p_t$, as the inflation factor. Then the excess demand of a young agent of type 1 is obtained from (2.b-1)

(2.g-1) $$f^1\left(\frac{1}{\pi_{t+1}}\right) = -\frac{\beta}{1+\beta},$$

since $R_t = p_t / p_{t+1}$. Similarly, the excess demand of a young agent of type 2 is

(2.g-2) $$f^2\left(\frac{1}{\pi_{t+1}}\right) = \frac{\pi_{t+1}}{(1+\beta)},$$

so that the aggregate excess demand function is

(2.g-3) $$f\left(\frac{1}{\pi_{t+1}}\right) = \frac{1}{1+\beta}\left[\pi_{t+1} - \beta\right].$$

In period 1, since the sum of excess demand should be zero, and the excess demand of the old is $2M / p_1$, $f(\pi_2) = -2M / p_1$. Thus, a competitive equilibrium is a sequence of allocations for each type $j$ $\left\{\left(x_t^{jt}, x_{t+1}^{jt}\right)\right\}_{t=1}^{\infty}$ and $\left\{\left(x_1^{j0}\right)\right\}$, $j$ = 1, 2, and inflation factors $\left\{(\pi_t)\right\}_{t=2}^{\infty}$, such that

**a)** $f(\pi_{t+1}) - \pi_t f(\pi_t) = 0, \quad \forall t \ge 1$,

**b)** $f\left(\frac{1}{\pi_t}\right) = \frac{1}{1+\beta}\left[\pi_t - \beta\right], \quad \forall t \ge 1$, and

**c)** $f(\pi_2) = -\frac{2M}{p_1}$.

Thus monetary equilibria satisfy the difference equation

(2.g-4) $$f(\pi_{t+1}) - \pi_t f(\pi_t) = 0 .$$

**2.h** A stationary equilibrium is one which satisfies conditions (i) – (iii) in (g) above, and in addition satisfies $\pi_t = \overline{\pi} \quad \forall t$ .

**2.i** If $\beta > 1$ there are two stationary equilibria in this economy, $\overline{\pi}_1 = \beta, \quad \overline{\pi}_2 = 1$. If $\beta = 1$ these two equilibria coincide and if $\beta < 1$, $\overline{\pi}_2$ is not an equilibrium. This is because to support the Golden Rule as an equilibrium would require the price level to be negative. At $t$ = 1, $f(\pi_2) = -\frac{2M}{p_1} < 0$, but from (2.g-3) we see that $f\left(\frac{1}{\overline{\pi}_2}\right) = \frac{1-\beta}{1+\beta} > 0$ if $\beta < 1$. Since prices are non-negative, $\overline{\pi}_2$ does not constitute a stationary equilibrium if $\beta < 1$.

**3.** This problem concerns an overlapping generations model with production.

**3.a** An *allocation* is a set of labor supplies and consumption sequences $\{n_t, c_{t+1}\}_{t=1}^{\infty}$ and a consumption for the initial young, $c_1$. A *feasible allocation* has the property that $c_t \le n_t$, for all t. A *Pareto efficient* allocation is a feasible allocation such that there exists no alternative allocation in which at least one has higher utility and every other agent has at least as much utility.

**3.b** If there is no money then all agents consume the fruit of their own production each period. Assuming no storage, this implies zero utility for every agent. A Pareto dominating allocation is one in which $n_t = 1/2$, $c_t = 1/2$, and $_t = 1/4$ for every generation other than the initial one. Utility of the initial old generation equals $/2$. This allocation would require each agent to work when young and consume when old.

**3.c** To restore Pareto efficiency, agents could pass money from one generation to another, beginning with the initial old. This "social contrivance" would support a stationary allocation with a real interest factor of unity. The price of commodities in terms of money would be sufficient to make the real value of the fixed stock of money equal to the real value of consumption each period.

**4.** This problem explores the standard two period overlapping generations model.

**4.a** A competitive equilibrium is a sequence of prices $\{p_t\}_{t=1}^{\infty}$ and a sequence of allocations $\left(\{x_t^t\}_{t=1}^{\infty}, \{x_{t+1}^t\}_{t=0}^{\infty}\right)$ such that, given these prices

- **a)** the allocation maximizes is the one that maximizes each agent's lifetime utility at equilibrium prices are:
- **b)** markets clear, i.e., $x_t^t + x_t^{t-1} = \omega^0 + \omega^1, \quad \forall t \ge 1.$

For generation $G_0$, the allocation is $x_1^0 = M / p_1$. Each member of generation $G_t, t \geq 1$, maximizes $U = c_t^t c_{t+1}^t$ subject to the budget constraint

$$c_t^t + (p_{t+1} / p_t) c_{t+1}^t = \omega^0 + (p_{t+1} / p_t)\omega^1 .$$

The excess demand of the young in period *t* is therefore

$$f(p_t / p_{t+1}) = \frac{1}{2}\left(\omega^0 - p_{t+1}\omega^1 / p_t\right). \tag{4.a-1}$$

Equilibrium is characterized by the following difference equation

$$f(p_t / p_{t+1}) + (p_{t-1} / p_t) f(p_{t-1} / p_t) = 0 . \tag{4.a-2}$$

It is easy to check that there are two stationary equilibria, $\overline{R}_1 = 1$ and $\overline{R}_2 = \omega^1 / \omega^0$, where $R_t = p_t / p_{t+1}$. Thus, corresponding to $\overline{R}_1$, the allocations are $c_t^t = (\omega^0 + \omega^1) / 2 = c_{t+1}^t$. The allocations corresponding to $\overline{R}_2$ are $c_t^t = \omega^0, c_{t+1}^t = \omega^1$. When the economy is Samuelson, $\overline{R}_1 > \overline{R}_2$, there is also a continuum of equilibria indexed by initial values of $R_t$ in the interval $R_0 = \dfrac{p_0}{p_1} \in \left[\overline{R}_2, \overline{R}_1\right]$.

**4.b** If there were no money in this economy, there would be no means of intergenerational trade, as the young would not lend to the old. The only equilibrium would be autarky. Such an equilibrium is Pareto-inefficient. A Pareto improving equilibrium would be defined by the stationary allocation that occurs at the Golden Rule.

**4.c** The assumption $\omega^0 > \omega^1$, implies that the economy is Samuelson. Hence an equilibrium with positively valued fiat money exists, and is Pareto-efficient.

**4.d** If $\omega^1 > \omega^0$, then the economy is Classical. The only equilibrium possible is the autarkic stationary equilibrium, which is Pareto-efficient in this case.