Chapter 7

1. To answer this question see Figure 7.2 on page 151 and the discussion on page 149 of the text.

2. This problem concerns a firm producing subject to increasing returns to scale.

2.a The least cost way of producing output $\overline{Y}$ is obtained from the minimization problem:

$$C\left(w,r,\overline{Y}\right) \equiv \min_{K,L} \quad wL + rK \quad + \ \tilde{\lambda}\ \left(\overline{Y} - AK^{\alpha}L^{\beta}\right) \ ,$$

which has three first order conditions:

(2.a-1) $$w - \frac{\tilde{\lambda}\beta\overline{Y}}{L} = 0,$$

(2.a-2) $$r - \frac{\tilde{\lambda}\alpha\overline{Y}}{K} = 0,$$

(2.a-3) $$AK^{\alpha}L^{\beta} = \overline{Y} .$$

Substituting (2.a-1)–(2.a-3) back into the objective function leads to the intermediate step

(2.a-4) $$C\left(w,r,\overline{Y}\right) = \theta\tilde{\lambda}\overline{Y},$$

where we define $\theta = \alpha + \beta$ to be the degree of returns to scale and $\tilde{\lambda}$ is the marginal cost of relaxing the constraint. Solving (2.a-1) for L, (2.a-2) for K and substituting the solutions into (2.a-3) leads to the solution for $\tilde{\lambda}$:

(2.a-5) $$\tilde{\lambda} = \left(\frac{r}{\alpha}\right)^{\frac{\alpha}{\theta}}\left(\frac{w}{\beta}\right)^{\frac{\beta}{\theta}}\overline{Y}^{\left(\frac{1-\theta}{\theta}\right)}A^{-\left(\frac{1}{\theta}\right)},$$

which can be substituted back into (2.a-4) to generate the cost function:

(2.a-6) $$C\left(\overline{Y}\right) = \gamma\overline{Y}^{\frac{1}{\theta}},$$

where γ is defined as:

(2.a-7) $$\gamma = \theta\left(\frac{w}{\beta}\right)^{\frac{\beta}{\theta}}\left(\frac{r}{\alpha}\right)^{\frac{\alpha}{\theta}}\left(\frac{1}{A}\right)^{\frac{1}{\theta}}$$

2.b Since $\theta > 1$ the cost function (2.a-6) (drawn in Figure 7.1) is concave in $\bar{Y}$.

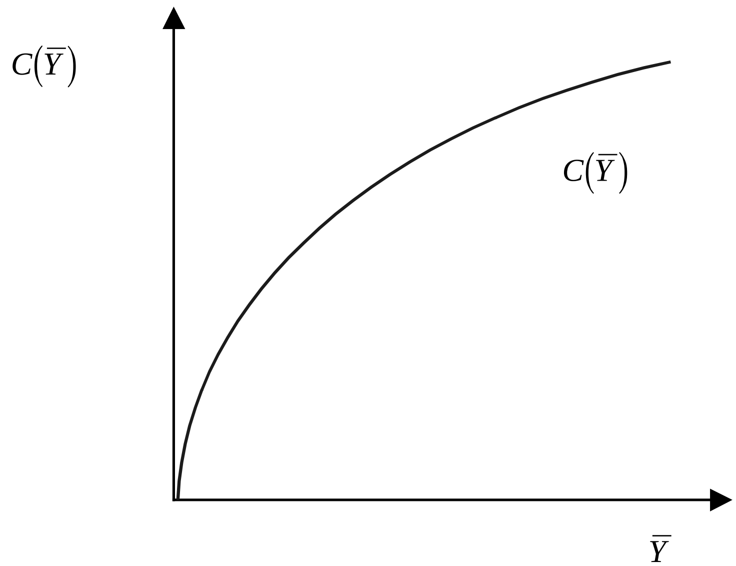

Figure 7.1

2.c The firm's profit function is

(2.c-1) $$\Pi(Y) = pY - C(Y) = Y^{1-\lambda} - \gamma Y^{1/\theta},$$

and the first order condition for condition for profit maximization is that:

(2.c-2) $$MR = (1-\lambda)Y^{-\lambda} = \gamma \frac{1}{\theta} Y^{\frac{1}{\theta}-1} = MC,$$

where MR stands for marginal revenue and MC for marginal cost. The value $Y*$ that satisfies this equation is given by the solution to:

(2.c-3) $$Y^{*\frac{1}{\theta}+\lambda-1} = \frac{(1-\lambda)\theta}{\gamma}.$$

Since $C(Y)$ is concave, $-C(Y)$ is convex, and it follows that for profits to be concave the revenue function $Y^{1-\lambda}$ must be sufficiently concave.

The necessary condition for concavity of Π is:

(2.c-4) $$\frac{d^2\Pi}{dY^2} = -\lambda(1-\lambda)Y^{-(1+\lambda)} - \gamma\frac{1}{\theta}\left(\frac{1}{\theta}-1\right)Y^{\frac{1}{\theta}-2} \leq 0,$$

which, rearranging terms, implies:

(2.c-5) $$-\lambda(1-\lambda) \le \gamma \frac{1}{\theta}\left(\frac{1}{\theta}-1\right)Y^{\frac{1}{\theta}+\lambda-1}.$$

Evaluated at Y^*, and using (2.c-3), this gives the condition:

(2.c-6) $$\lambda \ge \left(1-\frac{1}{\theta}\right),$$

which is a sufficient condition for the profit function to be locally concave implying that Y* is a maximum.

2.d Condition (2.c-4) implies that, the slope of the marginal revenue curve should be less than the slope of the marginal cost curve. Since both the marginal revenue and marginal cost curves are downward sloping, this implies that the marginal revenue curve has a steeper slope, as illustrated in Figure 7.2 below.

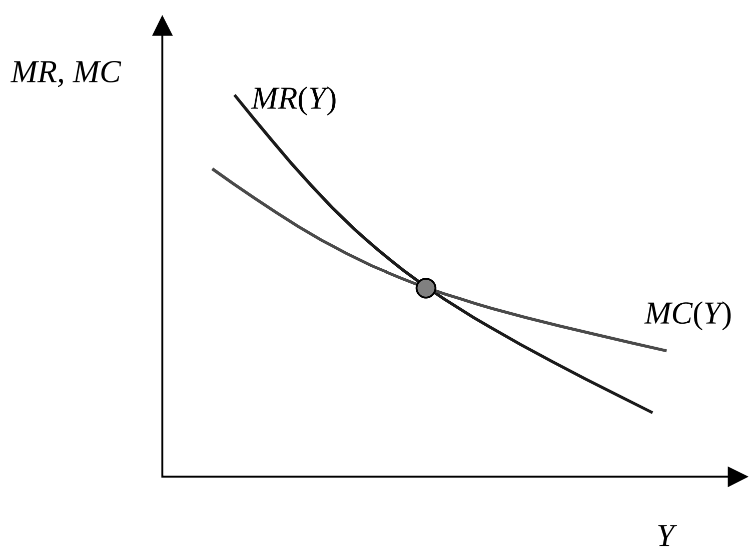

Figure 7.2

3. This problem concerns a real business cycle economy with increasing returns.

3.a The social production function is obtained by substituting for the externality factor, and imposing the equilibrium condition $\bar{K}_t = K_t, \bar{L}_t = L_t$. In this example it is given by

(3.a-1) $$Y_t = A_t\left(K_t\right)^{1/3}\left(L_t\right)^{2/3} = \left(K_t\right)^{2/3}\left(L_t\right)^{4/3}.$$

3.b The representative household maximizes $E_1 \sum_{t=1}^{\infty} \beta^{t-1} [\log C_t - L_t]$ subject to the period resource constraint $C_t = (1-\delta)K_t + AK_t^{1/3} L_t^{2/3} - K_{t+1}$. The first order conditions for this maximization problem are:

(3.b-1) $$C_t = \frac{2}{3}\left(\frac{Y_t}{L_t}\right),$$

(3.b-2) $$\frac{1}{C_t} = E_t\left[\frac{\beta}{C_{t+1}}\left\{1-\delta+\frac{1}{3}\left(\frac{Y_{t+1}}{K_{t+1}}\right)\right\}\right].$$

3.c The transversality condition is a first-order condition for optimality "at infinity". In words, it requires that the asymptotic value of the household's assets, weighted by its marginal utility of consumption should be zero:

(3.c-1) $$\lim_{t\to\infty} \beta^t u'(C_t) K_{t+1} = 0 \Rightarrow \lim_{t\to\infty} \beta^t \left(\frac{K_{t+1}}{C_t}\right) = 0.$$

3.d The steady state values of Y^*, C^*, K^* and L^* solve the following four equations:

(3.d-1) $$\delta K = Y - C$$

(3.d-2) $$\frac{1}{C} = \frac{\beta}{C}\left(1-\delta+\frac{1}{3}\frac{Y}{K}\right)$$

(3.d-3) $$C = \frac{2}{3}\left(\frac{Y}{L}\right), \text{ and}$$

(3.d-4) $$Y = K^{2/3} L^{4/3}.$$

First we solve for the ratio of *Y* to *K*. From (3.d-2),

(3.d-5) $$y^* \equiv \frac{Y^*}{K^*} = 3\left(\frac{1}{\beta} - 1 + \delta\right),$$

Now using (3.d-1) gives the ratio of *C* to *K*:

(3.d-6) $$c^* \equiv \frac{C^*}{K^*} = y^* - \delta.$$

Substituting these solutions into (3.d-3) gives the solution for L^*:

(3.d-7) $$L^* = \frac{2y^*}{3c^*}.$$

Now, from the production function (3.d-4):

(3.d-8) $$K^* = \frac{L^{*4}}{y^{*3}}.$$

Given K^* we can solve for Y^* and C^*:

(3.d-9) $$Y^* = y^* K^*, \qquad C^* = c^* K^*.$$

3.e Let $\tilde{X}_t = \left(X_t - X^*\right)/X^*$ denote the proportional deviation of X from its steady state. Using this definition the four equations that linearize the model are given by:

(3.e-1) $$\tilde{K}_{t+1} = (1-\delta)\tilde{K}_t + y^*\tilde{Y}_t - c^*\tilde{C}_t$$

(3.e-2) $$\tilde{Y}_t = \frac{2}{3}\tilde{K}_t + \frac{4}{3}\tilde{L}_t,$$

(3.e-3) $$\tilde{C}_t = \tilde{Y}_t - \tilde{L}_t,$$

(3.e-4) $$-\tilde{C}_t = -\tilde{C}_{t+1} + \frac{\beta y^*}{3}\tilde{Y}_{t+1} - \frac{\beta y^*}{3}\tilde{K}_{t+1}$$

3.f From (3.e-2) and (3.e-3),

(3.f-1) $$\tilde{Y}_t = 4\tilde{C}_t - 2\tilde{K}_t.$$

Using (3.f-1) in (3.e-1),

(3.f-2) $$\tilde{K}_{t+1} = (1-\delta-2y^*)\tilde{K}_t + \left(4y^* - c^*\right)\tilde{C}_t$$

Using (3.f-1) in (3.e-4), we obtain

(3.f-3) $$-\tilde{C}_t = \left(\frac{4\beta y^*}{3} - 1\right)\tilde{C}_{t+1} - \beta y^*\tilde{K}_{t+1}.$$

Therefore the system of first order difference equations is given by

(3.f-4) $$\begin{bmatrix} -1 & 0 \\ \left(4y^* - c^*\right) & 1-\delta-2y^* \end{bmatrix}\begin{bmatrix} \tilde{C}_t \\ \tilde{K}_t \end{bmatrix} = \begin{bmatrix} \left(\frac{4\beta y^*}{3} - 1\right) & -\beta y^* \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \tilde{C}_{t+1} \\ \tilde{K}_{t+1} \end{bmatrix}.$$

which can be rewritten as $\tilde{X}_t = J\tilde{X}_{t+1}$, where $\tilde{X}_t{}' = \left(\tilde{C}_t \quad \tilde{K}_t\right)'$, and

(3.f-5) $$J = \begin{bmatrix} -1 & 0 \\ (4y^* - c^*) & 1-\delta-2y^* \end{bmatrix}^{-1} \begin{bmatrix} \left(\frac{4\beta y^*}{3} - 1\right) & -\beta y^* \\ 0 & 1 \end{bmatrix}.$$

3.g Determinacy means that there is a locally unique equilibrium; a necessary and sufficient condition for the steady state to be indeterminate is that both roots of J are outside the unit circle. Since this model has large increasing returns to labor, the labor demand curve slopes up more steeply than the labor supply curve; a necessary and sufficient condition for indeterminacy in continuous time versions of this model. Since this is a discrete time model, this theorem does not apply and one would need to check computationally if the roots are indeed outside the unit circle.

4. This problem is similar to problem 3.

4.a The social production function represents the relationship between the social output and social inputs when all firms expand or contract together. The private production function is the relationship between the output and inputs of one firm holding constant the production of all other firms. These concepts will differ if there are externalities.

The social production function is obtained by substituting for the externality factor in the private production function, and recognizing that in a symmetric equilibrium production by each representative family is equal to the economywide average output. In this example:

(4.a-1) $$Y_t = A_t K_t^{\alpha} L_t^{1-\alpha} = Y_t^{\theta} K_t^{\alpha} L_t^{1-\alpha}$$
$$\Rightarrow Y_t = K_t^{\alpha/(1-\theta)} L_t^{(1-\alpha)/(1-\theta)}.$$

4.b The representative family maximizes

(4.b-1) $$U = \sum_{t=1}^{\infty} \beta^{t-1}\left[\log\left((1-\delta)K_t^i + Y_t^i - K_{t+1}^i\right) + \log\left(1-L_t^i\right)\right]$$

subject to

(4.b-2) $$Y_t^i = A_t\left(K_t^i\right)^{\alpha}\left(L_t^i\right)^{1-\alpha}.$$

The first order conditions are

(4.b-3) $$(1-\alpha)Y_t^i = C_t^i \frac{L_t^i}{1-L_t^i},$$

(4.b-4) $$\frac{1}{C_t^i} = \beta \frac{1}{C_{t+1}^i}\left(1-\delta+\alpha\frac{Y_{t+1}^i}{K_{t+1}^i}\right).$$

4.c For a Cobb-Douglas production function, labor and capital's shares of income are constants. For the production function $Y_t = A_t K_t^{\alpha} L_t^{1-\alpha}$, labor's share of GDP is $1-\alpha$. Since labor's share of GDP is observed to be equal to 2/3, therefore, $\alpha = 1/3$.

4.d An equilibrium is determinate if it is not indeterminate. Indeterminacy of an equilibrium in this context means that there is a continuum of solutions to the linearized system of difference equations. To establish indeterminacy of the equilibrium we follow a systematic approach: First, consider the linearized system of difference equations, where tildes denote proportionate deviations from the steady state and stars denote steady state values:

(4.d – 1) $$\tilde{K}_{t+1} = (1-\delta)\tilde{K}_t + \frac{Y*}{K*}\tilde{Y}_t - \frac{C*}{K*}\tilde{C}_t$$

(4.d – 2) $$\tilde{Y}_t = \frac{\alpha}{1-\theta}\tilde{K}_t + \frac{1-\alpha}{1-\theta}\tilde{L}_t$$

(4.d – 3) $$\tilde{Y}_t = \tilde{C}_t + \frac{1}{1-L*}\tilde{L}_t$$

(4.d – 4) $$-\tilde{C}_t = -\tilde{C}_{t+1} + \alpha\beta\frac{Y*}{K*}\tilde{Y}_{t+1} - \alpha\beta\frac{Y*}{K*}\tilde{K}_{t+1}$$

The required steady state values are obtained from the following system of equations:

(4.d – 5) $$\delta K* = Y* - C*$$

(4.d – 6) $$\frac{1}{C*} = \frac{\beta}{C*}\left(1-\delta+\alpha\frac{Y*}{K*}\right)$$

(4.d – 7) $$(1-\alpha)Y* = C*\frac{L*}{1-L*}$$

(4.d – 8) $$Y* = K*^{\frac{\alpha}{1-\theta}} L*^{\frac{1-\alpha}{1-\theta}}$$

Solve for $y* \equiv Y*/K*$ from (4.d-6):

$$(4.d-9) \qquad y^* = \frac{1}{\alpha}\left(\frac{1}{\beta} - 1 + \delta\right)$$

Using (4.d-5) gives the ratio $c^* \equiv C^*/K^*$.

$$(4.d-10) \qquad c^* = y^* - \delta$$

Finally, using (4.d-7):

$$(4.d-11) \qquad L^* = \frac{(1-\alpha)y^*}{c^* + (1-\alpha)y^*}$$

Next, the equations in the linearized dynamic system are classified into a group of static equations and a group of dynamic equations. We can see that (4.d-2) and (4.d-3) are static and that (4.d-1) and (4.d-4) are dynamic. We define $\tilde{K}_t$ and $\tilde{C}_t$ to be the state variables of the dynamic system, and the other two variables to be the co-state variables.

The system is reformulated as:

$$(4.d-12) \qquad \mathbf{A_1}\begin{pmatrix}\tilde{K}_t \\ \tilde{C}_t\end{pmatrix} + \mathbf{A_2}\begin{pmatrix}\tilde{Y}_t \\ \tilde{L}_t\end{pmatrix} = \begin{pmatrix}0 \\ 0\end{pmatrix}$$

$$(4.d-13) \qquad \mathbf{A_3}\begin{pmatrix}\tilde{K}_t \\ \tilde{C}_t\end{pmatrix} + \mathbf{A_4}\begin{pmatrix}\tilde{Y}_t \\ \tilde{L}_t\end{pmatrix} + \mathbf{A_5}\begin{pmatrix}\tilde{K}_{t+1} \\ \tilde{C}_{t+1}\end{pmatrix} + \mathbf{A_6}\begin{pmatrix}\tilde{Y}_{t+1} \\ \tilde{L}_{t+1}\end{pmatrix} = \begin{pmatrix}0 \\ 0\end{pmatrix}$$

$$(4.d-14) \qquad \mathbf{A_1} = \begin{bmatrix}\frac{\alpha}{1-\theta} & 0 \\ 0 & 1\end{bmatrix}$$

$$(4.d-15) \qquad \mathbf{A_2} = \begin{bmatrix}-1 & \frac{1-\alpha}{1-\theta} \\ -1 & \frac{1}{1-L^*}\end{bmatrix}$$

$$(4.d-16) \qquad \mathbf{A_3} = \begin{bmatrix}1-\delta & -c^* \\ 0 & 1\end{bmatrix}$$

$$(4.d-17) \qquad \mathbf{A_4} = \begin{bmatrix}y^* & 0 \\ 0 & 0\end{bmatrix}$$

$$(4.d-18) \qquad \mathbf{A_5} = \begin{bmatrix}-1 & 0 \\ -\alpha\beta y^* & -1\end{bmatrix}$$

$$(4.d-19) \qquad \mathbf{A_6} = \begin{bmatrix}0 & 0 \\ \alpha\beta y^* & 0\end{bmatrix}$$

Defining

$$(4.d-20) \qquad \mathbf{J} = -(\mathbf{A_3} - \mathbf{A_4}\mathbf{A_2}^{-1}\mathbf{A_1})^{-1}(\mathbf{A_5} - \mathbf{A_6}\mathbf{A_2}^{-1}\mathbf{A_1})$$

we can write the reduced form of the dynamic system as:

$$(4.d-21) \qquad \begin{pmatrix} \tilde{K}_t \\ \tilde{C}_t \end{pmatrix} = \mathbf{J} \begin{pmatrix} \tilde{K}_{t+1} \\ \tilde{C}_{t+1} \end{pmatrix}.$$

Indeterminacy of the equilibrium can now be established by checking the roots of **J**. If both roots of **J** are outside the unit circle the equilibrium is indeterminate.

```
/*********************************************************
                         Problem  7.4
                Finding a bifurcation point
                    t.h.zang, april 1999
**********************************************************/

new;
library pgraph;

/* starting value for theta */
theta=-1.91;

/* ending value for theta in the search */
endtheta=0.92;

/*step length */
step=0.001;

/* specification of parameters */
alpha=0.67;
beta=0.97;
delta=0.1;

/* solve for steady state values */
ystar=(1/alpha)*((1/beta)-1+delta);
cstar=ystar-delta;
lstar=(1-alpha)*ystar/(cstar+(1-alpha)*ystar);

/* initializing the output */
out1=1;
out2=0;
out3=0;
histtheta=theta;
out=out1~out2~out3;

DO WHILE theta<=endtheta;

/* define the matrices of the linearized system */
a1=zeros(2,2);
a1[1,1]=alpha/(1-theta);
a1[2,2]=1;

a2=zeros(2,2);
a2[1,1]=-1;
a2[1,2]=(1-alpha)/(1-theta);
a2[2,1]=-1;
a2[2,2]=1/(1-lstar);
```

```
a3=zeros(2,2);
a3[1,1]=1-delta;
a3[1,2]=-cstar;
a3[2,2]=1;

a4=zeros(2,2);
a4[1,1]=ystar;

a5=zeros(2,2);
a5[1,1]=-1;
a5[2,1]=-alpha*beta*ystar;
a5[2,2]=-1;

a6=zeros(2,2);
a6[2,1]=alpha*beta*ystar;


/* matrix in the reduced form system
  of stochastic difference equations for the state variables */
j=-inv(a3-a4*inv(a2)*a1)*(a5-a6*inv(a2)*a1);

/* diagonalize the system */
{ eval,evec }=eigv(j);     @find the eigenvalues and eigenvectors@

length=abs(eval);                 @taking the modulus@
sorted=sortc(length,1);

out=out|(1~(sorted'));
histtheta=histtheta|theta;   @the list of values for the plot@

theta=theta+step;

ENDO;


xy(histtheta,out);                @graphical output@
```