Chapter 8

1. This problem uses L'Hospital's rule.

1.a The function under consideration is:

(1.a-1)
$$(C, L) = \frac{\left\{ C \exp\left[-\frac{L^{1+\gamma}}{1+\gamma} \right] \right\}^{1-\sigma} - 1}{1-\sigma}$$

As $\sigma \to 1$ both the numerator and the denominator of this expression converge to zero. Using L'Hospitals rule we seek the ratio of the derivatives of the top and bottom. Taking the ratio

(1.a-2)
$$\underset{\sigma \to 1}{Lim} \left(\frac{\left\{ C \exp\left[-\frac{L^{1+\gamma}}{1+\gamma} \right] \right\}^{1-\sigma} - 1}{1-\sigma} \right) = \underset{\sigma \to 1}{Lim} \left(\frac{\partial N(\sigma) / \partial \sigma}{\partial D(\sigma) / \partial \sigma} \right),$$

where
$$N(\sigma) = \left\{ C \exp\left[-\frac{L^{1+\gamma}}{1+\gamma} \right] \right\}^{1-\sigma} - 1, \qquad \text{and} \qquad D(\sigma) = 1 - \sigma.$$

(1.a-3)
$$\frac{\partial N(\sigma)}{\partial \sigma} = -\left[\log(C) - \frac{L^{1+\gamma}}{1+\gamma} \right] \left\{ C \exp\left[-\frac{L^{1+\gamma}}{1+\gamma} \right] \right\}^{1-\sigma}, \qquad \frac{\partial D(\sigma)}{\partial \sigma} = -1,$$

(1.a-4)
$$\Rightarrow \underset{\sigma \to 1}{Lim} \left(\frac{\left\{ C \exp\left[-\frac{L^{1+\gamma}}{1+\gamma} \right] \right\}^{1-\sigma} - 1}{1-\sigma} \right) = \underset{\sigma \to 1}{Lim} \left(\frac{-\left[\log(C) - \frac{L^{1+\gamma}}{1+\gamma} \right] \left\{ C \exp\left[-\frac{L^{1+\gamma}}{1+\gamma} \right] \right\}^{1-\sigma}}{-1} \right),$$
$$= \ln(C) - \frac{L^{1+\gamma}}{1+\gamma}.$$

Q.E.D.

2. We seek an expression for the production possibilities frontier. The first step is to write the model in intensive form – equations (2.a-1)–(2.a-6) set up the notation that we need for this purpose.

2.a Define

(2.a-1) $$k_c = \frac{K_c}{L_c},\; k_I = \frac{K_I}{L_I},\; \lambda = \frac{L_c}{L},\; k = \frac{K}{L},$$

where K and L are the aggregate quantities of capital and labor. Then adding up constraints gives:

(2.a-2) $$\frac{L_I}{L} = (1-\lambda),$$

(2.a-3) $$\lambda k_c + (1-\lambda) k_I = k.$$

Using the definitions of the fraction of labor used in the consumption sector, λ, and of the sector specific capital labor ratios k_I and k_c we can write the production functions in each sector as follows:

(2.a-4) $$C = k_c^{\,a} \lambda L, \quad I = k_I^{\,m} (1-\lambda) L.$$

Substituting (2.a-4) into (2.a-3) leads to the expression:

(2.a-5) $$\left(\frac{C}{L}\right)^{1/a} \lambda^{\frac{a-1}{a}} + \left(\frac{I}{L}\right)^{1/m} (1-\lambda)^{\frac{m-1}{m}} = k.$$

We now turn to a statement of the problem. The production possibilities frontier is defined by the following condition:

(2.a-6) $$\max_{\lambda} C \;\; s.t. \;\; \left(\frac{C}{L}\right)^{1/a} \lambda^{\frac{a-1}{a}} + \left(\frac{\bar{I}}{L}\right)^{1/m} (1-\lambda)^{\frac{m-1}{m}} \le k.$$

In words, the production possibilities frontier delivers the maximum amount of consumption goods that can be obtained for a given amount of capital and labor as a function of the chosen output of investment goods, $\bar{I}$. The first order conditions for this problem yield the result:

(2.a-7) $$\left(\frac{C}{L}\right)^{1/a} \frac{a-1}{a} \lambda^{\frac{-1}{a}} = \left(\frac{\bar{I}}{L}\right)^{1/m} \frac{m-1}{m} (1-\lambda)^{\frac{-1}{m}}.$$

Using the facts that

(2.a-8) $$k_c = \left(\frac{C}{L\lambda}\right)^{1/a} \quad \text{and} \quad k_I = \left(\frac{\bar{I}}{L(1-\lambda)}\right)^{1/m}$$

from the production function definitions, equations (2.a-4), we can write the capital labor ratio in the consumption sector as a linear function of the capital labor ratio in the investment sector.

(2.a-9) $$k_c = \phi k_I, \quad \text{where} \quad \phi = \frac{m-1}{m}\frac{a}{a-1}.$$

Note that $a = m$, implies $\phi = 1$. Substituting (2.a-9) into (2.a-3) yields:

(2.a-10) $$k_I = \frac{k}{1-\lambda(1-\phi)}, \quad k_C = \frac{\phi k}{1-\lambda(1-\phi)}.$$

Substituting this result back into (2.a-8) we can write the production functions as follows:

(2.a-11) $$C = \left(\frac{\phi k}{1-\lambda(1-\phi)}\right)^a \lambda L, \qquad I = \left(\frac{k}{1-\lambda(1-\phi)}\right)^m (1-\lambda)L.$$

Equations (2.a-11) describe consumption and investment as functions of λ, as the economy moves along the frontier of the production possibilities frontier. When $a = m$, $\phi = 1$ and in this case:

(2.a-12) $$C = (\phi k)^a \lambda L, \qquad I = k^m (1-\lambda)L,$$

which implies that:

(2.a-13) $$C + I = k^a L.$$

Q.E.D.

3. This problem studies an RBC model with non-separable preferences.

3.a The required first order conditions are:

(3.a-1) $$-\frac{\partial U(C_t, L_t)/\partial L_t}{\partial U(C_t, L_t)/\partial C_t} = \frac{\partial Y_t}{\partial L_t}$$

for the choice of labor and

(3.a-2) $$\frac{\partial U(C_t, L_t)}{\partial C_t} = \beta E_t\left[\frac{\partial U(C_{t+1}, L_{t+1})}{\partial C_{t+1}}\left(1-\delta+\frac{\partial Y_{t+1}}{\partial K_{t+1}}\right)\right]$$

for the choice of capital. Given the functional forms of utility (as in problem 1) and the production function we can write these conditions as follows:

(3.a-3) $$C_t L_t^{\gamma} = (1-a)\frac{Y_t}{L_t},$$ for labor and

(3.a-4) $$\frac{\left[C_t \exp\left(-\frac{L_t^{1+\gamma}}{1+\gamma}\right)\right]^{1-\sigma}}{C_t} = \beta E_t\left[\frac{\left[C_{t+1}\exp\left(-\frac{L_{t+1}^{1+\gamma}}{1+\gamma}\right)\right]^{1-\sigma}}{C_{t+1}}\left(1-\delta+\frac{aY_{t+1}}{K_{t+1}}\right)\right]$$ for capital.

3.b The transversality condition is given by:

(3.b-1) $$\underset{T\to\infty}{Lim}\,\beta^T \frac{\partial U(C_T, L_T)}{\partial C_T} K_{T+1} = 0.$$

3.c A balanced growth path is an equilibrium of the model in which consumption, output and capital all grow at the same rate and labor supply is constant.

3.d Using the new definitions write the production function and the capital accumulation equation as follows:

(3.d-1) $$y_t = U_t k_t^{a} L_t^{1-a},$$

(3.d-2) $$(1+g)k_{t+1} = k_t(1-\delta) + y_t - c_t.$$

The first order conditions, (3.a-3) and (3.a-4) are given by:

(3.d-3) $$c_t L_t^{1+\gamma} = (1-a)y_t,$$

(3.d-4)
$$c_t^{-\sigma}\exp\left(-\frac{1-\sigma}{1+\gamma}L_t^{1+\gamma}\right) = \beta(1+g)^{-\sigma}E_t\left[c_{t+1}^{-\sigma}\exp\left(-\frac{1-\sigma}{1+\gamma}L_{t+1}^{1+\gamma}\right)\left(1-\delta+a\frac{y_{t+1}}{k_{t+1}}\right)\right].$$

3.e Set $e_t = U_t = 1$ for all *t*. Then along the balanced growth path the following steady state equations hold.

(3.e-1) $$y = k^a L^{1-a},$$

(3.e-2) $$k(g+\delta) = y - c,$$

(3.e-3) $$cL^{1+\gamma} = (1-a)y,$$

(3.e-4) $$= \beta(1+g)^{-\sigma}\left(1-\delta+a\frac{y}{k}\right).$$

To solve these equations follow the following steps:

i) Solve (3.d-4) for *y**/*k**.

ii) Solve (3.e-2) for *c**/*k**.

iii) Solve (3.d-3) for *L**.

iv) Solve (3.e-1) for *k**.

v) Use (i) and (ii) to find *y** and *c**.

3.f The equilibrium is determinate because the model satisfies all of the assumptions of standard general equilibrium theory.

3.g The first order condition for a representative household's choice of labor supply would be:

(3.g-1) $$C_t L_t^{1+\gamma} = \omega_t$$

where ω_t is the real wage. This equation defines the constant consumption labor supply curve.

To find the Frisch labor supply curve define:

(3.g-2) $$\lambda_t = C_t^{-\sigma}\left\{\exp\left[\frac{-L_t^{1+\gamma}}{1+\gamma}\right]\right\}^{1-\sigma}$$

to be the marginal utility of consumption. Using this expression write consumption as a function of λ_t:

(3.g-3) $$C_t = \lambda_t^{-\frac{1}{\sigma}} \left\{ \exp\left[\frac{-L_t^{1+\gamma}}{1+\gamma} \right] \right\}^{\frac{1-\sigma}{\sigma}}$$

Substituting this expression into the constant consumption labor supply curve gives the Frisch labor supply curve:

(3.g-4) $$L_t^{1+\gamma} \lambda_t^{-\frac{1}{\sigma}} \left\{ \exp\left[\frac{-L_t^{1+\gamma}}{1+\gamma} \right] \right\}^{\frac{1-\sigma}{\sigma}} = \omega_t .$$

Since the L.H.S. of (3.g-1) is increasing in L_t, the constant consumption labor supply curve, for these preferences, *cannot* slope down. The slope of the Frisch labor supply curve depends on the magnitude of σ since the L.H.S. of (3.g-4) may be decreasing in L_t when σ is small.

4. A complete solution of this problem will lead you to the current research frontier and we sketch only the outline of a solution here. Since g = 0, we deal here only with the solution with no growth.

4.a Equations (3.d-1) – (3.d-4) must be modified as follows:

(4.a-1) $$y_t = U_t k_t^{a(1+\theta)} L_t^{(1-a)(1+\theta)},$$

(4.a-2) $$k_{t+1} = k_t(1-\delta) + y_t - c_t .$$

The first order conditions, (3.a-3) and (3.a-4) are given by:

(4.a-3) $$c_t L_t^{1+\gamma} = (1-a) y_t ,$$

(4.a-4) $$c_t^{-\sigma} \exp\left(-\frac{1-\sigma}{1+\gamma} L_t^{1+\gamma} \right) = \beta E_t \left[c_{t+1}^{-\sigma} \exp\left(-\frac{1-\sigma}{1+\gamma} L_{t+1}^{1+\gamma} \right)\left(1-\delta + a\frac{y_{t+1}}{k_{t+1}} \right) \right],$$

The steady state solves:

(4.a-5) $$y = k^{a(1+\theta)} L^{(1-a)(1+\theta)},$$

(4.a-6) $$\delta k = y - c ,$$

(4.a-7) $$cL^{1+\gamma} = (1-a) y ,$$

(4.a-8) $$\frac{1}{\beta}=1-\delta+a\frac{y}{k}.$$

We now provide an algorithm to compute the steady state explicitly. Begin with (4.a-8) to solve for *y*/k*:*

(4.a-9) $$\frac{y^*}{k^*}=\left(\frac{1}{\beta}+\delta-1\right)\frac{1}{a} \quad = \quad \left(\frac{1}{0.95}+0.1-1\right)\frac{1}{0.33} \;=\; 0.46$$

and from (4.a-6):

(4.a-10) $$\frac{c^*}{k^*}=\frac{y^*}{k^*}-\delta \quad = \quad 0.46-0.1 \;=0.36.$$

Using (4.a-7) and using $\gamma=0$ gives:

(4.a-11) $$L=(1-a)\frac{y^*}{k^*}\frac{k^*}{c^*} \quad = \quad (1-0.33)\frac{0.46}{0.36} \;=\; 0.86.$$

Now from the production function solve for:

(4.a-12) $$k^*=\left(\frac{y^*}{k^*}\frac{1}{L^{*(1-a)(1+\theta)}}\right)^{\frac{1}{a(1+\theta)-1}} \quad = \quad \left(0.46\frac{1}{0.86^{(1-0.33)(1+\theta)}}\right)^{\frac{1}{0.33(1+\theta)-1}} \quad = \quad \frac{0.46^{\frac{1}{0.33\theta-0.67}}}{0.86^{\frac{0.67+0.67\theta}{0.33\theta-0.67}}}.$$

Finally

(4.a-13) $$y^*=\left(\frac{y^*}{k^*}\right)k^*, \qquad c^*=\left(\frac{c^*}{k^*}\right)k^*.$$

4.b To derive the log linear model, first define the parameter χ:

(4.b-1) $$\chi=-\frac{1-\sigma}{\sigma}L^{*\gamma+1},$$

then the log linear model is given by the following sets of equations equations. We begin with the three static equations of the model:

(4.b-2) $$\ln(C_t)=-\frac{1}{\sigma}\ln(\lambda_t)+\chi\ln(L_t).$$

(4.b-3) $$\ln(Y_t)=a(1+\theta)\ln(K_t)+(1-a)(1+\theta)\ln(L_t)+\ln(U_t),$$

(4.b-4) $\ln(C_t)+(1+\gamma)\ln(L_t)=\ln(1-a)+\ln(Y_t)$.

Putting together equations (4.b-2) – (4.b-4) gives the following linear system:

$$\text{(4.b-5)}\quad \underbrace{\begin{bmatrix} 0 & 1/\sigma & 0 \\ -a(1+\theta) & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix}}_{A_1} \underbrace{\begin{bmatrix} \ln(K_t) \\ \ln(\lambda_t) \\ \ln(U_t) \end{bmatrix}}_{X_t} + \underbrace{\begin{bmatrix} 0 & 1 & -\chi \\ 1 & 0 & -(1-a)(1+\theta) \\ -1 & 1 & (1+\gamma) \end{bmatrix}}_{A_2} \underbrace{\begin{bmatrix} \ln(Y_t) \\ \ln(C_t) \\ \ln(L_t) \end{bmatrix}}_{Z_t} = 0 .$$

Or,

(4.b-6) $Z_t = -A_2^{-1} A_1 X_t$.

We now derive the linearized dynamic equations:

(4.b-7) $\ln(K_{t+1}) = a_1 \ln(K_t) + a_2 \ln(Y_t) + a_3 \ln(C_t)$

(4.b-8) $\ln(\lambda_t) = \ln(\lambda_{t+1}) + r\ln(Y_{t+1}) - r\ln(K_{t+1}) + w_{t+1}$,

(4.b-9) $\ln(U_{t+1}) = \lambda \ln(U_t) + \ln(e_{t+1})$

The parameters a_1, a_2 and r are defined as:

$$a_1 = (1-\delta),\quad a_2 = \frac{y^*}{k^*},\quad a_3 = -\frac{c^*}{k^*},\quad r = \beta a\frac{y^*}{k^*}.$$

w_{t+1} is the expectational error in the Euler equation and hence defined as:

$$w_{t+1} = E_t\left[\ln(\lambda_{t+1}) + r\ln(Y_{t+1}) - r\ln(K_{t+1})\right] - \left[\ln(\lambda_{t+1}) + r\ln(Y_{t+1}) - r\ln(K_{t+1})\right].$$

From equations (4.b-7) – (4.b-9) we get the following linear system:

$$\underbrace{\begin{bmatrix} a_1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & \lambda \end{bmatrix}}_{A_3} \underbrace{\begin{bmatrix} \ln K_t \\ \ln \lambda_t \\ \ln U_t \end{bmatrix}}_{X_t} + \underbrace{\begin{bmatrix} a_2 & a_3 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{A_4} \underbrace{\begin{bmatrix} \ln Y_t \\ \ln C_t \\ \ln L_t \end{bmatrix}}_{Z_t}$$

(4.b-10)
$$+ \underbrace{\begin{bmatrix} -1 & 0 & 0 \\ -r & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}}_{A_5} \underbrace{\begin{bmatrix} \ln K_{t+1} \\ \ln \lambda_{t+1} \\ \ln U_{t+1} \end{bmatrix}}_{X_{t+1}} + \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ r & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{A_6} \underbrace{\begin{bmatrix} \ln Y_{t+1} \\ \ln C_{t+1} \\ \ln L_{t+1} \end{bmatrix}}_{Z_{t+1}}$$

$$+ \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}}_{A_7} \underbrace{\begin{bmatrix} \ln e_{t+1} \\ w_{t+1} \end{bmatrix}}_{\tilde{e}_{t+1}} = 0.$$

Putting together (4.b-5) and (4.b-10) gives the model:

(4.b-11)
$$X_t = J_1 X_{t+1} + J_2 \tilde{e}_{t+1},$$

where $J_1 = -\left[A_3 - A_4 A_2^{-1} A_1\right]^{-1}\left[A_5 - A_6 A_2^{-1} A_1\right]$, and determinacy depends on the roots of the matrix J_1.

4.c The computer code should

i) Find the balanced growth path
ii) Find the linearized coefficients, a_1, a_2 etc.
iii) Construct the matrices A_1, A_2 etc
iv) Construct the matrix J_1.
v) Find its roots.

4.d Part d repeats these steps for $\sigma = 2$.

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/*****************************************************************
                             Problem 8.4
                  Roots of a difference equation
                  Thomas Hintermaier, Sept. 1999
*****************************************************************/
new;

format 8,3;

/* defining the points for the externality
for which the system is evaluated */
points=seqa(0,0.01,100);

outlist=zeros(rows(points),7);      /* initialization of output */
```

```
z=1;
DO WHILE z<=rows(points);

/* specification of the parameters */
sigma=0.75;
beta=0.95;
a=0.33;
delta=0.1;
gam=0;
lam=0.007;

theta=points[z];


/* computing some steady state values and elasticitiy parameters */
yokss=(1/beta+delta-1)*1/a;
cokss=yokss-delta;
lss=(1-a)*yokss/cokss;
pa1=1-delta;
pa2=yokss;
pa3=-cokss;
pr=beta*a*yokss;
pchi=-(1-sigma)*(lss^(gam+1))/sigma;


/* definition of the matrices in the linearized dynamic system */
a1=zeros(3,3);
a1[1,2]=1/sigma;
a1[2,1]=-a*(1+theta);
a1[2,3]=-1;

a2=zeros(3,3);
a2[1,2]=1;
a2[1,3]=-pchi;
a2[2,1]=1;
a2[2,3]=-(1-a)*(1+theta);
a2[3,1]=-1;
a2[3,2]=1;
a2[3,3]=1+gam;

a3=zeros(3,3);
a3[1,1]=pa1;
a3[2,2]=-1;
a3[3,3]=lam;

a4=zeros(3,3);
a4[1,1]=pa2;
a4[1,2]=pa3;

a5=zeros(3,3);
a5[1,1]=-1;
a5[2,1]=-pr;
a5[2,2]=1;
a5[3,3]=-1;

a6=zeros(3,3);
```

```
a6[2,1]=pr;


/* defining the matrix j1 in the reduced dynamic system */
j1=-inv(a3-a4*inv(a2)*a1)*(a5-a6*inv(a2)*a1);

/* finding the eigenvalues of j1 and their modulus */
eval=eig(j1);
length=abs(eval);
@ print eval~length; @

IF z==1;
"theta           eval1                  eval2                 eval3
       modulus1      modulus2       modulus3";
ENDIF;

outlist[z,1]=theta;
outlist[z,2]=eval[1]; outlist[z,3]=eval[2]; outlist[z,4]=eval[3];
outlist[z,5]=length[1]; outlist[z,6]=length[2]; outlist[z,7]=length[3];

z=z+1;
ENDO;

outlist;
```