Chapter 9

1. The present value price of a state-dependent commodity is the price that must be paid in period 0, for delivery of a commodity at some future date, contingent on the state of nature. An example might be the price of an orange delivered in one years time if and only if there is a drought in Florida.

2. This problem illustrates the idea that market completeness depends on the rank of the payoff matrix.

2.a In this example, there are three states and three securities. The payoff matrix given by:

$$\begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \\ 3 & 5 & 7 \end{bmatrix}$$

has reduced rank and in this case markets are incomplete.

2.b If instead the payoff matrix is given by:

$$\begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \\ 0.5 & 1 & 1.5 \end{bmatrix}$$

then markets are also not complete, since again the payoff matrix does not have full rank.

3. For Arrow's formulation to be equivalent to Debreu's we require that:

a) Preferences are time consistent. This means that the preference ordering of an individual over all future state-contingent-choices is invariant to the resolution of uncertainty.

b) Markets must be complete.

4. Consider the case of three periods and two states in each period. Let π_A be the probability that state *A* occurs and π_B be the probability state *B* occurs. Assume further that these probabilities are independent and identical in each period. Let C_{ij}^3 be the consumption at date 3 if event *i* occurs at date 2 and event *j* at date 3 and let C_i^2 be consumption at date 2 if event *i* occurs at date 2.

4.a At date 1 the consumer maximizes (4.a-1):

$$\begin{aligned} &= \pi_A \pi_A \left[U\left(C_1, C_A^2, C_{AA}^3\right)\right] + \pi_A \pi_B \left[U\left(C_1, C_A^2, C_{AB}^3\right)\right] \\ &+ \pi_B \pi_A \left[U\left(C_1, C_B^2, C_{BA}^3\right)\right] + \pi_B \pi_B \left[U\left(C_1, C_B^2, C_{BB}^3\right)\right], \end{aligned} \tag{4.a-1}$$

where $$(x,y,z)=\left(x^{\lambda}+y^{\lambda}+z^{\lambda}\right)^{\frac{1}{\lambda}}.$$

Suppose that the relative price of goods C_{AA}^{3} and C_{AB}^{3} is equal to *p*. Then the consumer will satisfy the first order condition:

(4.a-2) $$\frac{\pi_A}{\pi_B}\frac{\partial U\left(C_1,C_A^2,C_{AA}^3\right)/\partial C_{AA}^3}{\partial U\left(C_1,C_A^2,C_{AB}^3\right)/\partial C_{AB}^3}=p.$$

Now suppose that state *A* occurs in period 2. If markets reopen in period 2 will the relative price of goods C_{AA}^{3} and C_{AB}^{3} still equal p? In period 2 the consumer maximizes:

(4.a-3) $$=\pi_A\left[U\left(C_A^2,C_{AA}^3\right)\right]+\pi_B\left[U\left(C_A^2,C_{AB}^3\right)\right],$$

where $$(y,z)=\left(y^{\lambda}+z^{\lambda}\right)^{\frac{1}{\lambda}}.$$

For arbitrary values of λ it will not be true that

(4.a-4) $$\frac{\pi_A}{\pi_B}\frac{\partial U\left(C_A^2,C_{AA}^3\right)/\partial C_{AA}^3}{\partial U\left(C_A^2,C_{AB}^3\right)/\partial C_{AB}^3}=\frac{\pi_A}{\pi_B}\frac{\partial U\left(C_1,C_A^2,C_{AA}^3\right)/\partial C_{AA}^3}{\partial U\left(C_1,C_A^2,C_{AB}^3\right)/\partial C_{AB}^3}=p.$$

In the special case $\lambda=1$, the ratios of marginal utilities are independent of C_1 and in this case (4.a-4) *will* hold.

4.b Equilibrium in Debreu's world would involve maximization of thew period 0 utility function subject to a set of state contingent commodity prices. In Arrow's world markets reopen at each date and, since preferences change as uncertainty unfolds, allocations will generally be different in equilibrium.

5. The number of commodities in this case is given by:

$$\sum_{t=1}^{70}2^{t-1}=\frac{1-2^{70}}{1-2}=2^{70}-1.$$

This is an *extremely* large number.

6. At date 2, a dollar for sure at date 3 can be obtained by buying one unit of each of the arrow securities. Its price is $Z_2^1 + Z_2^2$. (Note that in general these prices could also depend on the realization of the state in period 1.) At date 1, the price of a dollar for sure at date 2 is given by $Z_1^1 + Z_1^2$. It follows that at date 1 the price of a dollar for sure in date 3 is $Z_1^1\left(Z_2^1 + Z_2^2\right) + Z_1^2\left(Z_2^1 + Z_2^2\right)$.