Chapter 10

1. This question relates to Cass' and Shell's paper "Do Sunspots Matter?".

1.a In standard general equilibrium theory allocations and prices may differ across states of nature. Cass and Shell pose the following question: if preferences and endowments are identical across states, in an exchange economy, can there be an equilibrium in which allocations are different for at least one household? If the answer is yes then they say that sunspots matter. The result depends on the assumption that some agents are unable to participate in insurance markets.

1.b Incomplete markets means that the set of financial instruments is not rich enough to transfer wealth across all possible states of nature. Incomplete participation means that not all agents are able to participate in the securities markets.

1.c If there are complete markets and complete participation, the economy is identical to a standard Arrow-Debreu exchange economy. It follows from the first welfare theorem that all equilibria are Pareto optimal. A sunspot equilibrium cannot be Pareto optimal since risk averse agents would prefer a safe allocation to a random one – since all uncertainty is extrinsic; the safe allocation is feasible. Therefore a sunspot equilibrium cannot exist.

1.d Indeterminacy refers to a property of equilibria in general equilibrium models. An equilibrium is indeterminate if arbitrarily close to it, there exists another equilibrium. A sunspot equilibrium is one in which allocations differ across states even when all uncertainty is extrinsic. They are related only by the fact that in models with indeterminate equilibria it is often easy to construct examples of sunspot equilibria.

2. This problem concerns a simple stochastic overlapping generations economy.

2.a Since agents work only when young, there will be a positive demand for savings for all values of the interest rate. Since a Samuelson economy is one for which savings are positive at the golden rule interest rate; this economy is necessarily Samuelson.

2.b The budget constraints of the agent in each period of life are:

(2.b-1) $$p_t n_t = M, \qquad c_{t+1} p_{t+1} = M.$$

Putting these together gives the lifecycle constraint:

(2.b-2) $$c_{t+1} = n_t \frac{p_t}{p_{t+1}}.$$

2.c A competitive equilibrium is a price sequence $\{p_t\}_{t=1}^{\infty}$ and a sequence of allocations $\{n_t, c_t\}_{t=1}^{\infty}$ such that

i) markets clear in each period: $n_t = c_t = \dfrac{M}{p_t}$

ii) allocations are optimally chosen given the prices.

2.d A sunspot equilibrium is an equilibrium in which allocations vary across states even if uncertainty is extrinsic. To see if sunspot equilibria exist we first solve the consumer's problem:

(2.d-1)
$$\max \ E_t\left[\frac{\left(\frac{p_t}{p_{t+1}}n_t\right)^{1-\rho}}{1-\rho} - n_t\right].$$

The first order condition to this problem gives:

(2.d-2)
$$E_t\left[n_t^{-\rho}\left(\frac{p_t}{p_{t+1}}\right)^{1-\rho}\right] = 1.$$

Using the first period budget constraint:

(2.d-3)
$$\frac{p_t}{p_{t+1}} = \frac{p_t}{M}\frac{M}{p_{t+1}} = \frac{n_{t+1}}{n_t}.$$

Substituting this back into (2.d-2) it follows that any equilibrium sequence must obey the functional equation:

(2.d-4)
$$E_t\left[n_t^{-\rho}\left(\frac{n_{t+1}}{n_t}\right)^{1-\rho}\right] = 1 \Rightarrow n_t = E_t\left[(n_{t+1})^{1-\rho}\right].$$

Consider non-stochastic equilibria; these obey the equation

(2.d-5)
$$n_t = n_{t+1}^{1-\rho},$$

which has two steady states for $0<\rho<1$, $n=0$ and $n=1$; and one steady state, $n=1$, for $\rho>1$. The steady state $n=1$ is indeterminate for $\rho>2$ since in this case the slope of the difference equation $n_{t+1}=n_t^{\frac{1}{1-\rho}}$ satisfies the condition $-1<\left.\frac{\partial n_{t+1}}{\partial n_t}\right|_{n=1}<0$. It follows that one can construct sunspot equilibria in the class:

(2.d-6) $$n_{t+1}=\left[n_t\left(1+u_{t+1}\right)\right]^{\frac{1}{1-\rho}}.$$

where $E_t\left[u_{t+1}\right]=0$, for arbitrary u_{t+1} whenever $\rho>2$. To verify that this solution satisfies (2.d-4) notice that:

(2.d-7) $$E_t\left[\left(n_{t+1}\right)^{1-\rho}\right]=E_t\left\{\left[n_t\left(1+u_{t+1}\right)\right]^{\frac{1-\rho}{1-\rho}}\right\}=n_t.$$

3. Providing the process that generates beliefs remains stable, then yes it does make sense to estimate models of the economy in which sunspots matter. Further, these models can be used to predict.

4. This problem ties the sunspot literature back into the linear rational expectations models that we studied in the first part of the book.

4.a If x_{t+1} is i.i.d. then there will exist a unique rational expectations equilibrium if $|b|<1$. The rational expectations equilibrium is given by the degenerate function of x_t:

(4.a-1) $$y_t=\frac{a}{1-b}$$

4.b Now suppose that $x_{t+1}=\rho x_t+e_{t+1}$. In this case:

(4.b-1) $$\underset{z_t}{\begin{bmatrix} y_t \\ x_t \end{bmatrix}}=\underset{A}{\begin{bmatrix} b & b \\ 0 & 1/\rho \end{bmatrix}}\underset{z_{t+1}}{\begin{bmatrix} y_{t+1} \\ x_{t+1} \end{bmatrix}}+\underset{w_{t+1}}{\begin{bmatrix} a \\ e_{t+1} \end{bmatrix}}.$$

Uniqueness requires the roots of A to split around unity; these roots are b and $/\rho$. Since we assume the x_t is stationary ρ must be less than one in absolute value and hence uniqueness once again requires that $|b|<1$. The function that characterizes the unique equilibrium is given by:

(4.b-2) $$y_t=\frac{-q^{12}}{q^{11}}x_t$$

where q^{11} and q^{12} are elements of the first row of Q^{-1}, where Q is the matrix of eigenvectors of A.

4.c If $|b| > 1$, there may be multiple stationary rational expectations equilibria which are generated as solutions to the equation:

(4.c-1) $$z_{t+1} = A^{-1} z_t - A^{-1} w_{t+1}$$

where

(4.c-2) $$w_{t+1} = \begin{bmatrix} a u_{t+1} \\ e_{t+1} \end{bmatrix}$$

and u_{t+1} is an arbitrary random variable with mean equal to unity. In this case u_t is, by definition, a sunspot.