A SUNSPOT-BASED THEORY OF UNCONVENTIONAL MONETARY POLICY

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Abstract. This paper is about the effectiveness of qualitative easing, a form of unconventional monetary policy that changes the risk composition of the central bank balance sheet with the goal of stabilizing economic activity. We construct a general equilibrium model where agents have rational expectations and there is a complete set of financial securities, but where some agents are unable to participate in financial markets. We show that a change in the risk composition of the central bank's balance sheet will change equilibrium asset prices and we prove that, in our model, a policy in which the central bank stabilizes non-fundamental fluctuations in the stock market is Pareto improving and self-financing.

Central banks throughout the world have recently engaged in two kinds of unconventional monetary policies: *quantitative easing* (QE), which is "an increase in the size of the balance sheet of the central bank through an increase in its monetary liabilities", and *qualitative easing* (QualE) which is "a shift in the composition of the assets of the central bank towards less liquid and riskier assets, holding constant the size of the balance sheet."[1]

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An earlier incarnation of this paper with the title "Qualitative Easing: How it Works and Why it Matters" (Farmer, 2012d) appeared as an NBER and a CEPR working paper. The current version was written after extensive discussions with Pawel Zabczyk during, and following, Farmer's visit, as Senior Houblon-Norman Fellow at the Bank of England in 2013. In contrast to the earlier paper, here we introduce money. Farmer thanks the Center for Central Bank Studies at the Bank of England for their hospitality. We thank participants of seminars at the National Institute for Economic and Social Research, Warwick University, The Swiss National Bank and the Bank of Italy. Bernard Winkler, Francesco Lippi and Christian Hepenstrick have all contributed thoughtful dicussions of our work and we thank them for their comments. We have also benefited from discussions with Andy Haldane, Minouche Shafik and Martin Weale. The views expressed in this paper are those of the authors, and not necessarily those of the Bank of England.

[1]The quote is from Willem Buiter (2008) who proposed this very useful taxonomy in a piece on his 'Maverecon' Financial Times blog. Farmer (2013) used this distinction to argue that the Bank should actively stabilize the asset markets. This paper provides the theory that explains why Qualitative Easing is effective. Earlier working papers that explain why financial markets may be Pareto Inefficient include Farmer (2002b,a, 2012c, 2014, 2015) and Farmer et al. (2012).

Because qualitative easing is conducted by the central bank, it is often classified as a monetary policy. But because it adds risk to the public balance sheet that is ultimately borne by the taxpayer, QualE is better thought of as a fiscal or quasi-fiscal policy (Buiter, 2010). This distinction is important because, in order to be effective, QualE necessarily redistributes resources from one group of agents to another.

In theoretical papers that study the effectiveness of QualE, researchers often assume that financial markets are complete and that there is complete participation in these markets. When these two conditions hold, a change in the risk composition of the central bank's balance sheet has no effect on asset prices.

For example, in an influential paper that was presented at the 2012 Jackson Hole Conference, Michael Woodford (2012) made the claim that QualE is unlikely to be effective and, to the extent that it does stimulate economic activity, that stimulus must come through the impact of QualE on the expectations of financial market participants of future Fed policy actions.[2] In contrast, Joseph Gagnon (2012) has argued that qualitative easing works through the *portfolio balance effect*, a term attributed to James Tobin (1963; 1969) who assumed that private agents' asset demands are functions of relative asset prices, much as the demands for commodities depend on relative goods prices.[3]

QualE is ineffective when participation in assets markets is complete because market participants are able to undo the effects of a portfolio shift by the central bank through private trades in securities. As a consequence, QualE has no effect on the distribution of resources, either between borrowers and lenders in the current financial markets, or between current market participants and those yet to be born. We will argue here, that the assumption of complete participation is not a good characterization of real world financial markets and that QualE is effective because it redistributes resources across states of nature for people who are unable to participate in financial markets that open before they are born.

We make the case for the effectiveness of qualitative easing by constructing an analytically tractable general equilibrium model where agents are rational and have rational expectations and where the financial markets are complete. Our setup has two important features. First, some people in our model do not trade in the financial markets. This means that, while our results have a flavour similar to those in Arajo et al. (2015), which studies the efficacy of unconventional monetary policy under endogenous collateral constraints, they naturally generalize to setups such as the overlapping generations model (OLG). There, despite the *dynamic-completeness* of asset markets and the absence of any

[2]Citing papers by Krugman (1998) and Eggertsson and Woodford (2002) where the case is made explicitly, Woodford (2012) argues that this so-called portfolio balance view is invalid, and, if central bank asset purchases are to be effective, their effectiveness must rely on their ability to alter the public's expectations of future central bank policies.

[3]Examples of recent empirical papers that find a significant effect of Fed asset purchases on asset prices include Vissing Jorgensen and Krishnamurthy (2011); Gagnon et al. (2011); D'Amico and King (2010); Neely (2010); Li and Wei (2012), and Hamilton and Wu (2012).

collateral constraints, incomplete participation follows from the fact that people have finite lives. Second, people in our model use money as a medium of exchange. This result ensures that, in the absence of uncertainty, the model possesses multiple equilibria. It also implies that the underlying mechanism is different from that in Arajo et al. (2015), and can be seen as an application of the insights of Cass and Shell (1983) to the study of QualE effectiveness in a monetary economy.

In this environment, we show that 1) a central bank that takes risk onto its balance sheet can increase welfare and 2) the optimal intervention restores efficiency and is self-financing. When all uncertainty is non-fundamental, the optimal policy is for the central bank to stabilize the stock market so that the return to the stock market is equal in every state to the return on a one-period real government bond. In the presence of fundamental shocks, the government intervention eliminates non-fundamental volatility.

1. How our Model is Related to Previous Literature

Our model is related to the work of Cass and Shell (1983). These authors construct a two-period, purely real, pure-exchange general-equilibrium model. In the first period, of their model, households trade financial assets. In the second period they trade goods. In the Cass-Shell example there are multiple equilibria in the second period. They show that, if some households are not present in period 1, purely non-fundamental uncertainty can influence the equilibrium allocation of goods across households.

We adapt the Cass-Shell example in two ways. First, we introduce money as a medium of exchange. Second, we build a model with production, rather than pure exchange. Adding money allows us to explain the distinction between conventional monetary policy, which alters the size of the central bank's balance sheet, and unconventional monetary policy, which alters its composition between safe and risky assets. Adding production, allows us to explain how unconventional monetary policy can alter output and employment.

To model money, we include the real value of money balances as an argument of utility functions. This approach originated with Don Patinkin (1956) and we think of it as a short-cut that explains why people choose to hold an asset that is dominated in rate-of-return. For convenience, and to simplify algebra, we use the money wage as our numeraire and we assume that money, measured relative to the money wage, yields utility.[4]

Frank Hahn (1965) pointed out that monetary general equilibrium models always contain at least two equilibria; one in which money has value and one in which it does not. In these models, money is not essential for exchange, and there is always an equilibrium in which the price level is infinite and exchange is accomplished by barter.

[4]It is common in general equilibrium models of money, to divide cash-balances by the price level. We divide instead by the money wage to simplify some of the algebra. Nothing of substance in our argument hinges on this modeling choice.

In addition to these two equilibria, one with barter and one where money has value, infinite horizon general equilibrium monetary models typically also contain a continuum of non-stationary equilibria. Sometimes, as shown by Brock (1974), most of these equilibria are unrealistic as a description of reality: They converge to a steady state where money has no value. But in other examples, as shown by Benhabib et al. (2001), there is a continuum of equilibria that converge to a steady state where money has positive value but the money interest rate is low. These equilibria provide a possible description of the current situation where interest rates in many western economies have been equal to zero, or even negative, for several years.

It is possible to construct dynamic examples of the argument we make in this paper.[5] To keep our argument as transparent as possible, we have chosen instead to use a two-period model. That presents the challenge of explaining why the people in our model would choose to hold an asset, fiat money, that will be worthless when the model ends. To meet that challenge, we adopt a device proposed by Starr (1974). We assume that money is required to pay taxes at the end of the second period. Money has value in our model because the government decrees it to be so.

Related work of which we are aware includes papers by Michael McMahon and Herakles Polemarchakis (2011), McMahon, Udara Peiris and Polemarchakis (2015) and Peiris and Polemarchakis (2015). Although the environments they study are similar to ours, these authors do not study the optimal monetary policy and they do not explicitly model an equilibrium selection rule as we do here. Robert Hall and Ricardo Reis 2016 have studied the implications of policies that pay interest on reserves for price level stabilization and Ricardo Reis 2016 studies the role of unconventional monetary policies in response to a future fiscal crisis. Neither of these approaches considers the implications, for monetary policy, of incomplete asset market participation.

Two alternative theories to ours include the market segmentation approach of Dimitri Vayanos and Jean-Luc Vila (2009) and Robin Greenwood, Samuel Hanson and Gordon Liao (2015). These authors posit that different asset purchasers inhabit different segments of the market. Alternatively, Mark Gertler and Peter Karadi (2011), Vasco Curdia and Michael Woodford (2011) and Zhiguo He and Arvind Krishnamurthy (2013) present theories in which capital constraints may be alleviated by large scale central bank asset purchases that offset the restrictions imposed by borrowing restrictions. Krishnamurthy and Annette Vissing Jorgensen (2011) survey these alternative approaches and discuss the evidence in favour of each. They conclude that intervention in the MBS market had bigger effects than purchases of long-bonds, a conclusion that is further substantiated by evidence of Marco Di Maggio, Amir Kermani and Christopher Palmer (2015).

Our own work is complementary to both the segmentation approach of Vayanos and Vila (2009) and Greenwood et al. (2015) and the capital constraints theories of Gertler

[5] See for example, Farmer (2016).

and Karadi (2011), Curdia and Woodford (2011) and He and Krishnamurthy (2013). We provide a *deep* theory of market segmentation that explains why some asset market participants are constrained in their ability to participate in some markets. Those markets open before the relevant people are born.

2. A simple Two-Period Model

In this section we construct a simple stylized model that is, nevertheless, rich enough to capture the main points of our argument.

2.1. Assumptions about Workers and Entrepreneurs.

There are two periods, three types of people and two public agents; a central bank and a treasury. We refer to type 1 and 2 people as workers and to type 3 people as entrepreneurs. Workers are alive in both periods and they are each endowed, in period 2, with one unit of leisure. Entrepreneurs are alive only in period 2. They are endowed with a technology for producing a unique consumption good in period 2. The fact that we call these people 'entrepreneurs' is not important to the economic arguments we will make. In a more complicated model with multiple periods and long lives there would be people of all types present in all periods.

In period 1, workers trade in asset markets with each other, with the central bank, and with the treasury. Production and consumption takes place only in period 2. There is a paper asset called money, that is an argument of workers' utility functions. Workers face the following budget constraint in period 1,

$$M_i + QA_i - TR_i = 0,$$

were, TR_i is a nominal transfer to workers by the treasury. This money transfer can be held as money, M_i, or interest bearing bonds, A_i. Q is the period 1 price of a dollar-denominated pure discount bond and the subscript $i \in \{1, 2\}$ indexes workers.

In period 2, workers face the constraint,

$$pc_i + w(1 - n_i) \leq w + M_i + A_i - \mathcal{T}_i.$$

Here, w is the money wage, p is the price of commodities, n_i is labor supply, c_i is consumption and $\mathcal{T}_i$ is a lump-sum tax obligation. Putting together the budget constraints of workers for periods 1 and 2, and rearranging terms, leads to the life-cycle constraint,

$$pc_i + w(1 - n_i) + rM_i \leq \mathcal{W}_i, \tag{1}$$

where

$$\mathcal{W}_i \equiv w + \frac{TR_i}{Q} - \mathcal{T}_i, \tag{2}$$

is the dollar value, at date 2, of a worker's wealth and

$$r \equiv \frac{1 - Q}{Q},$$

is the money interest rate.

2.2. **Assumptions about the Treasury and the Central Bank.** The treasury pays for its transfer to people in period 1 by issuing dollar denominated debt,

$$QB = \sum_i TR_i \equiv TR. \tag{3}$$

The left side of this equation is the value of debt floated by the treasury in the asset markets and the right side is the value of the dollar denominated transfers to the private sector. The value of debt is broken into the face value of dollar denominated pure discount bonds, denoted by B, and their price at date 1, denoted by Q. We refer to the choice of B as a *fiscal policy.*

An amount A_{CB} of treasury debt is purchased by the central bank and the remaining portions A_1 and A_2 are purchased by workers. This leads to the following asset market clearing equation,

$$QB = QA_{CB} + QA_1 + QA_2.$$

The portion of the debt purchased by the central bank is equal to the monetary base. Because we do not include private banks in our model, the monetary base is equal to the money supply and all of it is held by private agents as checking accounts at the central bank,

$$QA_{CB} = M.$$

We refer to the choice of M as a *monetary policy.*

Because the bank does not pay interest on its liabilities, the creation of money generates equity for the central bank equal to the present value of the bank's seigniorage revenues,

$$E_{CB} = Q\mathcal{S},$$

where $\mathcal{S}$ is defined as,

$$\mathcal{S} \equiv (A_{CB} - M) = rM.$$

Table 1 represents the balance sheet of the central bank in period 1.

Assets	Liabilities
QA_{CB}	M
$Q\mathcal{S}$	
	E_{CB}

TABLE 1. The Central Bank Balance Sheet

At date 2, the treasury must repay its debt by raising taxes $\mathcal{T}$ on the private sector, or from seigniorage revenues, $\mathcal{S}$, received from the central bank,

$$B = \mathcal{T} + \mathcal{S}. \tag{4}$$

Replacing B in Equation (3) from Equation (4) leads to the intertemporal government budget constraint,

$$Q(\mathcal{T} + \mathcal{S}) = TR.$$

This equation clarifies that the dollar value of the transfer to the workers in period 1 is equal to the present value of tax revenues plus the present value of seigniorage revenue.

2.3. **The Equal Treatment Assumption.** We assume that people alive in each period are treated equally, and thus each worker receives half the transfer

$$TR_1 = TR_2 = \frac{TR}{2},$$

and workers and the entrepreneurs each pay a third of the tax burden,

$$\mathcal{T}_1 = \mathcal{T}_2 = \mathcal{T}_3 = \frac{\mathcal{T}}{3}. \tag{5}$$

We can then express workers' wealth $\mathcal{W}_i$ in equation (2) as

$$\mathcal{W} \equiv w + \left(\frac{B}{6} + \frac{rM}{3}\right), \tag{6}$$

where the equal treatment assumptions imply that $\mathcal{W}$ is identical for both workers, allowing us to drop the subscript i.[6] This expression makes clear that the value of the transfer to workers depends on both fiscal policy, represented by B, and monetary policy, represented by M.

Fiscal policy in our model is a pure transfer from one generation to another. All government expenditures in period 1 are in the form of transfer payments to workers. Those transfers are repaid in period 2 by taxes levied on both workers and entrepreneurs and from seigniorage revenues. Because entrepreneurs are not present in period 1, they do not benefit from transfers. They do, however, incur part of the cost of paying for those transfers.[7]

3. Equilibria Under the Perfect Foresight Assumption

In this section we derive the demand and supply functions of workers and entrepreneurs and we define the concept of a competitive perfect foresight equilibrium.

[6]In words, Equation (6) says that the period 2 money value of the wealth of a young worker is equal to the money value of his leisure endowment plus 1/6 of the period 2 value of his transfer plus 1/3 of the government's seigniorage revenue from money creation. The fraction 1/6 appears because workers receive half of the government transfer but only have to repay a third of it ($1/6 = 1/2 - 1/3$). The fraction 1/3 of seigniorage revenue follows from the fact that, for a given transfer, additional seigniorage revenues reduce the tax burden on all three types. More generally, the wealth effect of a transfer policy will depend on the population growth rate and the period length.

[7]It would be relatively simple to extend our model to allow for government purchases and for trade in goods in period 1. We have not pursued that extension because our goal in this paper is to focus on the role of trade in the asset markets. In our view, central bank asset trades have non-trivial effects because they generate fiscal transfers between generations that cannot be undone by private markets. We show subsequently that these trades can sometimes be Pareto improving.

Our main result is that, because the transfer is denominated in nominal units, there is a different perfect foresight equilibrium for every value of the numeraire.

3.1. **The Behavior of Workers.** Workers have logarithmic preferences defined over consumption, leisure and the real value of money balances in period 2 with weights λ_i on consumption, μ_i on leisure and γ_i, on real money balances defined in wage units,[8]

$$U_i = \lambda_i \log c_i + \mu_i \log(1-n_i) + \gamma_i \log\left(\frac{M_i}{w}\right),$$

where

$$\lambda_i + \mu_i + \gamma_i = 1.$$

Workers differ in their preference weights and we use the subscript i to index type. It is not essential for our main results that there should be two types of workers. We make this assumption to ensure that our model has a non-trivial equilibrium where there is active trade in the asset markets.

Workers maximize utility subject to the lifecycle constraint, Inequality (1). The solution to this problem, given the assumption of logarithmic preferences, is for the workers' expenditure shares on leisure, consumption and money to equal the respective coefficients in the utility function times wealth, $\mathcal{W}$, that is,

$$w(1-n_i) = \mu_i \mathcal{W}, \tag{7}$$

$$pc_i = \lambda_i \mathcal{W},$$

$$rM_i = \gamma_i \mathcal{W}. \tag{8}$$

Rearranging terms in Equation (7), we obtain the following expression for the labor supply function of type i,

$$n_i = 1 - \mu_i \frac{\mathcal{W}}{w}.$$

3.2. **The Behavior of Entrepreneurs.** Entrepreneurs do not participate in the asset markets since, by assumption, they are born in period 2. Each entrepreneur owns a decreasing returns-to-scale technology,

$$y = n^\alpha,$$

that transforms labor into output. Entrepreneurs receive real profits, Π, defined as,

$$\Pi \equiv n^\alpha - \frac{w}{p} n,$$

and their consumption is subject to the constraint

$$c_3 \leq \Pi - \frac{\mathcal{T}_3}{p}.$$

[8]The assumption of logarithmic utility is not important for our results. All the qualitative properties of our model can be derived for the case of general concave utilities.

Using the equal treatment assumption, Equation (5), and the consolidated government budget constraint, Equation (4), we may write taxes, $\mathcal{T}_3$ as,

$$\mathcal{T}_3 = \frac{B - rM}{3}.$$

Entrepreneurs choose c_3 and n to solve the problem

$$\max_{\{c_3, n\}} U_3 = \log(c_3),$$

such that $c_3 \leq \Pi - \frac{\mathcal{T}_3}{p}$. The solution is characterized by the labor demand function,

$$n = \left(\frac{1}{\alpha}\frac{w}{p}\right)^{\frac{1}{\alpha-1}}, \tag{9}$$

the output supply function,

$$y = \left(\frac{1}{\alpha}\frac{w}{p}\right)^{\frac{\alpha}{\alpha-1}},$$

and the entrepreneur's consumption demand function,

$$c_3 = (1-\alpha)\left(\frac{1}{\alpha}\frac{w}{p}\right)^{\frac{\alpha}{\alpha-1}} - \frac{(B - rM)}{3p}. \tag{10}$$

We have assumed that entrepreneurs have logarithmic utility. That assumption is not important for the construction of a perfect foresight equilibrium. As long as their utility is increasing in consumption, entrepreneurs will maximize profit Π. But the assumption that preferences are logarithmic implies that entrepreneurs and workers are risk averse: this property will affect outcomes when we discuss equilibria under uncertainty in Section 5. Because people are risk averse, equilibria where non-fundamental shocks influence the allocation of goods across states are Pareto inefficient.

3.3. **The Definition of Perfect Foresight Equilibria.** In this section we write down three equations that characterize equilibria. These are excess demand equations for labor, consumption and money. It follows from Walras law that only two of these equations are independent of each other and thus, if we find prices that solve two of them, the third equation will also be satisfied.

If this were a standard Arrow-Debreu general equilibrium model (Arrow and Debreu, 1954), we would be able express the excess demand functions in terms of relative prices. But in our model, there is a dollar denominated transfer from one group of people to another. It follows that the price level is not simply a numeraire; it determines the real value of the nominal transfer. We will show that there is a different equilibrium for every value of the numeraire. In each of these equilibria it is not not only money prices that differ, but also employment, output and the distribution of output between generations.

To resolve the indeterminacy of equilibria, following (Farmer, 1999), we introduce a new fundamental, the belief function. The belief function acts as a selection device that determines the behavior of people in an environment where there is more than one equilibrium action.

The three goods in our model are labor, consumption, and money. The three dollar denominated prices are the money price of goods, p, the money wage, w and the money interest rate, r. We will characterize equilibria as the solutions to the excess demand functions for two of these three goods.

To simplify our analysis we first define three new parameters,

$$\lambda \equiv \sum_{i \in \{1,2\}} \lambda_i, \qquad \mu \equiv \sum_{i \in \{1,2\}} \mu_i, \qquad \gamma \equiv \sum_{i \in \{1,2\}} \gamma_i. \tag{11}$$

Using these definitions, the labor market clearing equation is,

$$\overbrace{\left(\frac{1}{\alpha}\frac{w}{p}\right)^{\frac{1}{\alpha-1}}}^{\text{Labor Demand}} = \overbrace{2 - \mu\frac{\mathcal{W}}{w}}^{\text{Labor Supply}}, \tag{12}$$

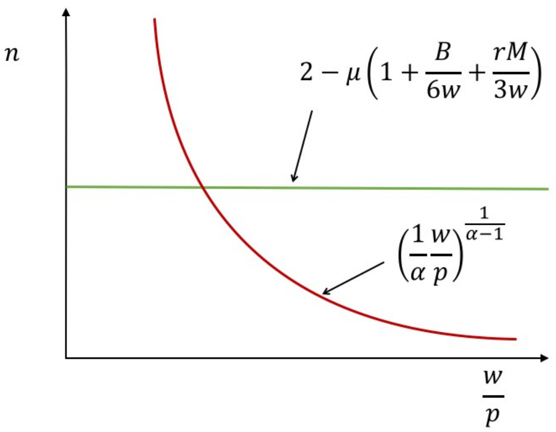
the goods market clearing equation is,

$$\overbrace{\left(\frac{1}{\alpha}\frac{w}{p}\right)^{\frac{\alpha}{\alpha-1}}}^{\text{Output}} = \overbrace{(1-\alpha)\left(\frac{1}{\alpha}\frac{w}{p}\right)^{\frac{\alpha}{\alpha-1}} - \frac{(B - rM)}{3p}}^{\text{Entrepreneur's Consumption}} + \overbrace{\frac{w}{p}\lambda\frac{\mathcal{W}}{w}}^{\text{Workers' Consumption}}, \tag{13}$$

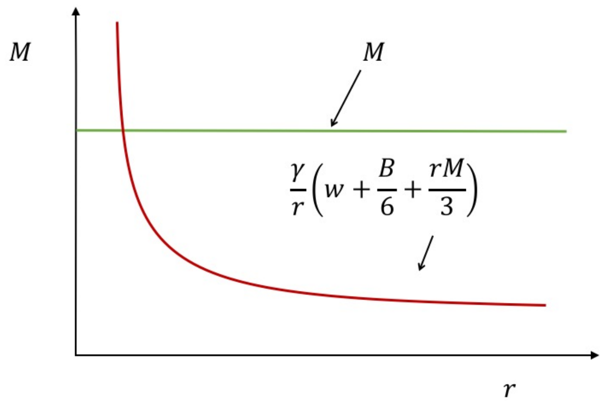
and the money market clearing equation is,

$$\overbrace{\gamma\frac{\mathcal{W}}{r}}^{\text{Money Demand}} = \overbrace{M}^{\text{Money Supply}}, \tag{14}$$

where $\mathcal{W}$ is the value of workers' wealth, defined in Equation (6).



(A) Labor Market Clearing.



(B) Money Market Clearing.

FIGURE 1. The Demand and Supply of Labor and Money.

Figure 1(A) plots the demand and supply for labor as functions of the real wage, the money wage and the money interest rate. The real wage appears on the horizontal axis and the quantities of labor demanded and supplied are on the vertical axis. The downward sloping curve is the labor demand function; this is the left-hand-side of Equation (12). The horizontal line is the labor supply function; this is the right-hand-side of Equation (12). The function is horizontal because, when people have logarithmic preferences, the income and substitution effects exactly balance each other.

Figure 1(B) plots the demand and supply of money as functions of the money wage, w, and the money interest rate r. The horizontal line is the money supply. The downward sloping curve is the demand for money as a function of r.

The following definition of a competitive equilibrium is fairly standard. What sets our model apart from more familiar general equilibrium models without money is that the value of w appears independently in the labor market clearing equation and in the money market clearing equation.

Definition 1. *An equilibrium is a monetary policy and a fiscal policy* $\{M, B\}$, *an allocation* $\{\{n_i, M_i\}_{i\in 1,2}, \{c_i\}_{i=1,2,3}\ , y, n\}$ *and a set of prices* $\{p, w, r\}$ *that satisfies non-negativity, budget balance and optimality. An equilibrium price system is a non-negative triple* $\{p, w, r\}$ *such that equations (12), (13) and (14) hold.*

Proposition 1 establishes that there is a continuum of equilibria and it characterizes them in closed form.

Proposition 1. *Let* $\{M, B\} \geq 0$ *characterize monetary and fiscal policy, and let* $w > 0$ *satisfy the feasibility conditions,*

$$w \geq \frac{\mu_i B}{4(1+\lambda+\mu) - 6\mu_i}, \quad i \in \{1,2\} \quad \text{and} \quad w \geq \frac{2-\mu+\lambda(2-3\alpha)}{\mu+\alpha\lambda}\frac{B}{2}.$$

The equilibrium level of nominal wealth, the interest rate and the real wage are given by,

$$\mathcal{W} = \frac{6w+B}{2(1+\lambda+\mu)}, \quad r = \frac{\gamma}{M}\mathcal{W}, \qquad \frac{w}{p} = \alpha\left(2 - \frac{\mu\mathcal{W}}{w}\right)^{\alpha-1}. \tag{15}$$

The equilibrium values of $\{\{n_i, M_i\}_{i\in 1,2}, \{c_i\}_{i=1,2,3}\ , y, n\}$ *are determined by equations (7) – (8) and (9) – (10) respectively.* $\square$

See Appendix A for a proof of this proposition.

In our model, there is a continuum of equilibria because transfers are defined in nominal units but they have real effects on the allocation of resources between workers and entrepreneurs. Although we have proven this in a two period model, the result is more general.[9]

3.4. **Beliefs and Equilibrium Selection.** In any model with multiple equilibria we must take a stand on what selects an equilibrium. Multiplicity occurs in a dynamic model because there is more than one possible future price. A decision maker, placed in

[9]Every general equilibrium model of money contains multiple equilibria since there is always at least one equilibrium where money has value and a second equilibrium where it does not (Hahn, 1965). Generically, models in this class also contain a continuum of non-stationary equilibria for which the inflation rate converges to the rate associated with one of the stationary equilibria.

an environment where many different things can happen, must take an action. Following Farmer (2012b), we select what will happen by modeling the way people form beliefs as a new primitive of the model with the same methodological status as preferences and technology.

In the 1960s, it was common to distinguish the future value of a price, we will call this P, from the expectation of that price, we will call this P^E. Because P and P^E were modeled as separate objects the researcher was obliged to introduce a new equation to explain how P^E is determined. Following the work of Friedman (1957), macroeconomists often assumed that expectations are formed adaptively. Under the adaptive expectations hypothesis,

$$P_{t+1}^E = \psi P_t^E + (1 - \psi) P_t,$$

where ψ is a parameter that determines the speed of adjustment of the belief about the future price to its actual value and the subscript t denotes time period.

With the introduction of rational expectations by Robert Lucas (1972), the way that economists model expectations changed. Lucas argued that the world is uncertain and that the people who inhabit our models would be expected to adapt to uncertainty. He suggested that prices fluctuate because of random shocks to the fundamentals of the economy. If the economy enters some state, captured by the random variable X, the price we observe will be a function, $P(X)$. If X is stationary, every time the world enters state X the people who inhabit our model would expect to observe the same price, $P(X)$. Eventually, they will learn the mapping from X to $P(X)$ and they will form their expectations $P^E(X)$ using the equilibrium price function $P(X)$.

That argument was widely accepted and for the past forty years, almost all macroeconomic models have adopted the rational expectations assumption. The argument is persuasive; but it relies on the assumption that the rational expectations equilibrium is unique. In monetary models, the uniqueness property is almost always violated. In models with multiple rational expectations equilibria we cannot dispense with an equation that explains how expectations are formed. Farmer (2002c) referred to that equation as the belief function and we will adopt that same terminology here.

The belief function, $\varphi(\cdot)$, is a mapping from present and past observable variables to a probability measure over future prices. For example, the people in our model might believe that,

$$w^E = \varphi(B, M, \varepsilon),$$

where w^E is the anticipated money wage next period and ε is a random variable with probability distribution,

$$\varepsilon = \begin{cases} H & \text{with probability} \quad \pi_H \\ L & \text{with probability} \quad \pi_L. \end{cases}$$

Here, ε captures the possibility that people may over-react or under-react to fundamental uncertainty as well as the possibility that they may react to purely non-fundamental shocks. If expectations are rational, the model will be closed by the equation,

$$w^E(\varepsilon) = w(\varepsilon), \quad \varepsilon \in \{H, L\}.$$

In words, the realization of the money wage, $w^E(\varepsilon)$, that people expect to occur if the random variable ε is realized, will coincide with $w(\varepsilon)$, the value of w that actually occurs.

Consider the following simple example of a belief function. People believe that w may take one of two values,

$$\varphi(M, \varepsilon) = M + \varepsilon \equiv \begin{cases} M + H \equiv w(H) & \text{with probability} \quad \pi_H \\ M + L \equiv w(L) & \text{with probability} \quad \pi_L. \end{cases} \tag{16}$$

We have included the policy variable M in the belief function to capture the idea that beliefs depend on observable variables. In this example, beliefs are independent of B but they do depend on M. We have included the random variable ε in the belief function to capture the idea that non-fundamental uncertainty may matter simply because people believe that it will. We will demonstrate in the next section that non-fundamental beliefs cause allocations to fluctuate even when there is a complete set of insurance markets.[10]

4. Introducing Uncertainty to the Model

In this section we expand the model to allow for non-fundamental uncertainty. We assume that the workers assume, correctly, that there are two possible future realizations of the money wage. In one state of the world, state H, the money wage is high and in the other, state L, it is low.

We allow workers to write complete insurance contracts that pay one amount in state H and another in state L. We show that, because the entrepreneurs cannot participate in the insurance market, employment, output and the allocation of consumption across states is different in different states. Our result is an example of the Cass and Shell (1983) result that, when there is incomplete participation, 'sunspots matter'.

4.1. **Budget Constraints Under Uncertainty.** We model uncertainty with the assumption that the money wage in Period 2 may take one of two values and that people form beliefs using Equation (16). If $\varepsilon = H$, the realization of the money wage is equal to $w(H)$ and if $\varepsilon = L$ the realization of the money wage is equal to $w(L)$. Further, these beliefs are correct. The fact that the money wage will be different in different states influences the asset market trades that workers make in period 1.

There are complete insurance markets, represented by a pair of Arrow securities (Arrow, 1964), one for each state. The H security pays one dollar if and only if state H

[10]Our example, where money is the only fundamental that affects beliefs, is very special. More generally, beliefs about the future wage might depend on current and past wages, or on current and past output or employment. For an example of a complete model closed with a belief function see Farmer (2012a).

occurs and the L security pays one dollar if and only if state L occurs. The H security costs $Q(H)$ dollars in period 1 and the L security costs $Q(L)$ dollars. Because there are complete markets, workers face a single life-cycle budget constraint. They may transfer income across dates and across states by buying or selling these securities.

In period 1, each worker receives a transfer TR_i that he may use to acquire money M_i, and he may buy or sell Arrow securities $A_i(\varepsilon)$,

$$\sum_{\varepsilon \in \{H,L\}} Q(\varepsilon) A_i(\varepsilon) + M_i \leq TR_i.$$

In period 2 state ε, the worker faces the constraint,

$$p(\varepsilon) c_i(\varepsilon) + w(\varepsilon)(1 - n_i(\varepsilon)) \leq w(\varepsilon) + A_i(\varepsilon) + M_i - \mathcal{T}_i.$$

Substituting for $A_i(\varepsilon)$ from the period 2 constraint into the first period gives the following lifecycle budget constraint,

$$\sum_{\varepsilon \in \{H,L\}} Q(\varepsilon)\left[p(\varepsilon) c_i(\varepsilon) + w(\varepsilon)(1 - n_i(\varepsilon)) - w(\varepsilon) - M_i + \mathcal{T}_i\right] + M_i \leq TR_i.$$

To make the connection between this model, which has two dates and two states, with a standard general equilibrium model with one date and multiple goods, it will help if we introduce the concept of *state prices.* The state prices of leisure and consumption are the dollar costs of a unit of leisure or a unit of consumption for delivery in Period 2 in state ε. These prices are denominated in units of Period 1 dollars and they are divided by the probability that state ε will occur. Dividing by probabilities in this way allows us to replace the summation operator in the budget constraint by an expectations operator.

The state prices of leisure and consumption are defined as,[11]

$$\tilde{w}(\varepsilon) \equiv \frac{Q(\varepsilon)}{Q}\frac{w(\varepsilon)}{\pi_\varepsilon} \quad \text{and} \quad \tilde{p}(\varepsilon) \equiv \frac{Q(\varepsilon)}{Q}\frac{p(\varepsilon)}{\pi_\varepsilon}. \tag{17}$$

We may also exploit the no-arbitrage assumption to find a connection between Q, $Q(H)$ and $Q(L)$

$$\sum_{\varepsilon \in \{H,L\}} Q(\varepsilon) = Q,$$

Using these definitions we may write the lifecycle budget constraint of a worker as follows,

$$\mathbb{E}\left[\tilde{p}(\varepsilon) c_i(\varepsilon) + \tilde{w}(\varepsilon)(1 - n_i(\varepsilon))\right] + rM_i \leq \mathcal{W}, \tag{18}$$

where workers' wealth in the model with uncertainty, but complete markets, is defined as

$$\mathcal{W} \equiv \mathbb{E}[\tilde{w}(\varepsilon)] + \left(\frac{B}{6} + \frac{rM}{3}\right). \tag{19}$$

Here $\mathbb{E}$ is the expectations operator, defined using the probability distribution $\{\pi_H, \pi_L\}$ and the term in the last bracket denotes the net transfer from the government.

[11]Note that this definition implies $\tilde{w}(\varepsilon)/\tilde{p}(\varepsilon) = w(\varepsilon)/p(\varepsilon)$ which we shall exploit subsequently.

4.2. **Workers' Choices Under Uncertainty.** Next, we turn to the behavior of workers in our expanded model where there are two possible future realizations of the money wage and the money price. Workers maximize the following logarithmic expected utility function,

$$\mathcal{U}_i = \mathbb{E}\left[\lambda_i \log\left(c_i\left(\varepsilon\right)\right) + \mu_i \log(1 - n_i\left(\varepsilon\right)) + \gamma_i \log\left(\frac{M}{w\left(\varepsilon\right)}\right)\right],$$

subject to the lifecycle budget constraint, Equation (18). The assumption that preferences are logarithmic implies, as in the certainty case, that budget shares are constant,

$$\tilde{w}\left(\varepsilon\right)\left(1 - n_i\left(\varepsilon\right)\right) = \mu_i \mathcal{W}, \tag{20}$$

$$\tilde{p}\left(\varepsilon\right) c_i\left(\varepsilon\right) = \lambda_i \mathcal{W}, \tag{21}$$

$$r M_i = \gamma_i \mathcal{W}. \tag{22}$$

Rearranging terms in Equation (20) leads to the labor supply function for type i in state ε,

$$n_i\left(\varepsilon\right) = 1 - \mu_i \frac{\mathcal{W}}{\tilde{w}\left(\varepsilon\right)}.$$

If $w\left(H\right)$ is different from $w\left(L\right)$, the real value of the worker's net transfer will differ across states. One might think that this difference would be irrelevant in a complete markets environment, with only non-fundamental uncertainty.

People, by assumption, are risk averse, and they would prefer the mean of a gamble to the gamble itself. Because all uncertainty is non-fundamental, that mean is available. Why don't people remove this welfare reducing uncertainty through trade? For two people to engage in a mutually beneficial insurance trade they must both be present before the random event occurs. Those trades cannot take place in our environment because entrepreneurs are not present to buy or sell Arrow securities.

4.3. **Entrepreneurs' Choices Under Uncertainty.** We have derived five equations that characterize how workers behave under uncertainty when markets are complete. There are two state-dependent equations for labor supply, two state-dependent equations for consumption demand, and one non-state-dependent equation for money demand. Next, we turn to the behavior of entrepreneurs.

Entrepreneurs are not present in the first period and they are unable to trade Arrow securities. This assumption implies they must solve two different problems, one for each realization of ε. In state ε, the entrepreneurs solve the problem,

$$\max_{\{c_3(\varepsilon), n(\varepsilon)\}} U_3\left(\varepsilon\right) = \log(c_3\left(\varepsilon\right)),$$

subject to the budget constraint

$$c_3\left(\varepsilon\right) \leq \Pi\left(\varepsilon\right) - \frac{\mathcal{T}_3}{p\left(\varepsilon\right)}.$$

The term $\Pi(\varepsilon)$, represents real profits in state ε and is defined as follows,

$$\Pi(\varepsilon) \equiv (n(\varepsilon))^{\alpha} - \frac{\tilde{w}(\varepsilon)}{\tilde{p}(\varepsilon)} n(\varepsilon).$$

The term $\mathcal{T}_3$, defined as

$$\mathcal{T}_3 \equiv \frac{(B - rM)}{3},$$

is the money value of the entrepreneur's tax liability. The solution to the entrepreneurs' problems leads to the labor demand functions,

$$n(\varepsilon) = \left(\frac{1}{\alpha}\frac{\tilde{w}(\varepsilon)}{\tilde{p}(\varepsilon)}\right)^{\frac{1}{\alpha-1}}, \tag{23}$$

the output supply functions,

$$y(\varepsilon) = \left(\frac{1}{\alpha}\frac{\tilde{w}(\varepsilon)}{\tilde{p}(\varepsilon)}\right)^{\frac{\alpha}{\alpha-1}}, \tag{24}$$

and the entrepreneurs' consumption demand functions,

$$c_3(\varepsilon) = (1-\alpha)\left(\frac{1}{\alpha}\frac{\tilde{w}(\varepsilon)}{\tilde{p}(\varepsilon)}\right)^{\frac{\alpha}{\alpha-1}} - \frac{\tilde{w}(\varepsilon)}{\tilde{p}(\varepsilon)}\frac{(B - rM)}{3w(\varepsilon)}. \tag{25}$$

Notice that $w(\varepsilon)$ enters Equation (25) independently of $\tilde{w}(\varepsilon)$. That follows from the fact the dollar value of taxes paid by entrepreneurs is the same in both states, but it's real value depends on the realization of the money wage and the money price of goods.

5. Incomplete Participation Equilibrium in the World of Uncertainty

In Sections 5, 6 and 7, we explore the properties of equilibria in a world of uncertainty under three different assumptions about the environment. We begin, in Section 5, by defining an equilibrium in a world where there is a complete asset market but entrepreneurs cannot participate in this market. We show, in this world, that non-fundamental uncertainty may have real effects on the output produced and on its allocation across people.

In Section 6, we ask the counter-factual question: what would the equilibrium look like if we did allow entrepreneurs to participate in the asset market? We show, in this environment, that output, employment and consumption allocations are the same in both states. Finally, in Section 7 we show that government can restore the complete participation equilibrium by trading on behalf of the entrepreneurs.

5.1. **Rational Expectations Equilibrium with Incomplete Participation.** The model with two states has five goods: consumption in states H and L, leisure in states H and L and money. There are four state-contingent prices, $\tilde{p}(H)$, $\tilde{p}(L)$, $\tilde{w}(H)$, $\tilde{w}(L)$, and one non state-contingent interest rate, r.

Entrepreneurs' consumption demand functions are given by Equation (25). These equations depend, not just on state prices, but also on the dollar wages, $w(H)$ and $w(L)$. This fact is the key to understanding why non-fundamental uncertainty has real effects.

Using the definitions of λ and μ from Equation (11), the labor market clearing equation in state ε is given by the expression,

$$\left(\frac{1}{\alpha}\frac{\tilde{w}(\varepsilon)}{\tilde{p}(\varepsilon)}\right)^{\frac{1}{\alpha-1}} = 2 - \mu\frac{\mathcal{W}}{\tilde{w}(\varepsilon)}. \tag{26}$$

The goods market clearing equation in state ε is found by equating output supply from Equation (24) to the sum of workers' consumption demands, from Equation (21), and entrepreneurs' consumption demands, from Equation (25). Rearranging terms and dividing by $\tilde{w}(\varepsilon)/\tilde{p}(\varepsilon)$ we arrive at Equation (27),

$$\left(\frac{1}{\alpha}\frac{\tilde{w}(\varepsilon)}{\tilde{p}(\varepsilon)}\right)^{\frac{1}{\alpha-1}} = \lambda\frac{\mathcal{W}}{\tilde{w}(\varepsilon)} - \left(\frac{B - rM}{3w(\varepsilon)}\right). \tag{27}$$

The equality of the demand and supply of money, gives one additional equation,

$$M = \gamma\frac{\mathcal{W}}{r}. \tag{28}$$

To complete our characterization of a rational expectations equilibrium, we define the belief function,

$$w(\varepsilon) = \varphi(M, \varepsilon) \equiv M + \varepsilon. \tag{29}$$

This is a fundamental equation of our model that represents the way people form their beliefs. In a rational expectations equilibrium, these beliefs are not only fundamental, they are also rational in the sense that they are correct in equilibrium. Using the definitions of the market clearing equations, we define an incomplete participation rational expectations equilibrium as follows,

Definition 2. *An incomplete participation rational expectations equilibrium is a pair of monetary and fiscal policies* $\{M, B\}$*, a belief function* $\varphi(M, \varepsilon)$*, an allocation* $\{\{n_i(\varepsilon), M_i\}_{i\in 1,2}, \{c_i(\varepsilon)\}_{i=1}$ $y(\varepsilon), n(\varepsilon)\}_{\varepsilon\in\{H,L\}}$*, a set of state-prices* $\{\tilde{p}(\varepsilon), \tilde{w}(\varepsilon), r\}_{\varepsilon\in\{H,L\}}$*, and a pair of Arrow security prices* $\{Q(\varepsilon)\}_{\varepsilon\in\{H,L\}}$ *that satisfies budget balance and optimality.*

An equilibrium price system is a non-negative 7-tuple $\{\tilde{p}(\varepsilon), \tilde{w}(\varepsilon), Q(\varepsilon), r\}_{\varepsilon\in\{H,L\}}$ *such that equations (26) and (27) hold in each state* ε*, Equation (28) holds and the money wage in each state is given by the belief function, Equation (29).*

We now turn to a proposition that characterizes the properties of an incomplete participation rational expectations equilibrium. Because we assume that people have rational expectations, the belief function determines the money wage in each state, $w(L), w(H)$. For any given money wage, the market clearing equations determine $\tilde{w}(\varepsilon), \tilde{p}(\varepsilon), Q(\varepsilon)$ and r and, as in the perfect foresight case, the feasibility condition rules out belief functions that would result in negative prices or negative allocations in one or more states.

Proposition 2. *Let $\{M, B\} \geq 0$ characterize public sector policy, and let $\{w(L), w(H)\} > 0$ be wages implied by a belief function $\varphi(M, \varepsilon)$. Assume further that the feasibility condition holds in both states*

$$w(\varepsilon) \geq \frac{\mu_i B}{4(1+\lambda+\mu) - 6\mu_i}, \quad i \in \{1,2\}, \quad \varepsilon \in \{L, H\} \tag{30}$$

$$w(\varepsilon) \geq \frac{2-\mu+\lambda(2-3\alpha)}{\mu+\alpha\lambda}\frac{B}{2}. \quad \varepsilon \in \{L, H\} \tag{31}$$

Define the following constants $\theta, X_L, Y_L, X_H, Y_H, \theta_1$ and θ_2,

$$\theta \equiv \frac{\lambda+\mu}{\gamma},$$

$$X_L \equiv [6w(L) + B], \quad X_H \equiv [6w(H) + B], \quad Y_L \equiv 3\pi_L\theta, \quad Y_H \equiv 3\pi_H\theta,$$

$$\theta_1 \equiv \frac{[X_H(1+Y_L) - X_L(1+Y_H)]}{X_L Y_H}, \quad \theta_2 \equiv \frac{X_H Y_L}{X_L Y_H}.$$

The equilibrium ratio of Arrow security prices $q \equiv Q(L)/Q(H)$, is the unique positive solution to the quadratic equation

$$q^2 - \theta_1 q - \theta_2 = 0. \tag{32}$$

The equilibrium Arrow security prices satisfy

$$Q(H) = \frac{(q + Y_L[q+1])M}{X_L q(1+q) + (1+q)(q+Y_L[q+1])M}, \tag{33}$$

and

$$Q(L) = qQ(H). \tag{34}$$

The equilibrium state wages $\tilde{w}(\varepsilon)$ are given by the expressions

$$\tilde{\omega}(L) = \frac{w(L)}{(1+q^{-1})\pi_L}, \quad \tilde{\omega}(H) = \frac{w(H)}{(1+q)\pi_H}, \tag{35}$$

and the state prices are equal to

$$\tilde{p}(\varepsilon) = \frac{\tilde{w}(\varepsilon)}{\alpha}\left(2 - \frac{\mu\mathcal{W}}{\tilde{w}(\varepsilon)}\right)^{1-\alpha}, \quad \varepsilon \in \{L, H\}, \tag{36}$$

where

$$Q = Q(L) + Q(H), \quad r \equiv \frac{1-Q}{Q}, \quad \text{and} \quad \mathcal{W} = \frac{(1-Q)}{Q}M \quad . \tag{37}$$

Conditional on the $\tilde{w}(\varepsilon), \tilde{p}(\varepsilon)$ and $Q(\varepsilon)$ characterized above, the equilibrium quantities $c_i(\varepsilon), n_i(\varepsilon)$ and M_i can then be determined from Equations (20) – (22), while $n(\varepsilon), y(\varepsilon)$ and $c_3(\varepsilon)$ are characterized in Equations (23) – (25).

Proposition 2, proved in Appendix B, establishes a mapping from beliefs to equilibrium prices and allocations. The following corollary confirms that these beliefs don't only affect nominal prices but also the corresponding real allocations.

Corollary 3. *Whenever $w(L) \neq w(H)$,*

$$n_i(L) \neq n_i(H), \quad \text{and} \quad c_i(L) \neq c_i(H). \tag{38}$$

Corollary 3 is proved in Appendix C. This is an example, for a monetary economy, of Cass and Shell's (1983) result that, when there is incomplete asset market participation, "sunspots matter".

6. Complete Participation Equilibrium in the World of Uncertainty

In this section we consider a counter-factual economy in which entrepreneurs are present in the asset markets that open before they are born and we derive their decision rules in the asset markets. In Sub-section 6.2 we will use this result to characterize a rational expectations equilibrium with complete participation and we will show that, in this equilibrium, output, employment and allocations are stabilized across states. Although there are still multiple equilibria in the complete participation case, sunspots cease to have real effects. We will also show that for a given belief function, which we take as primitive, the resulting equilibrium is Pareto efficient.

6.1. Entrepreneur's Choice under Complete Participation.

We continue to assume that entrepreneurs only care about consumption and receive no part of the government transfer. We alter the assumptions of the previous section by allowing entrepreneurs to trade assets in Period 1, subject to the first period constraint,

$$\sum_{\varepsilon \in \{L,H\}} Q(\varepsilon) A_3(\varepsilon) = 0. \tag{39}$$

The entrepreneur's maximum pre-tax profit in each state is given by,

$$\Pi(\varepsilon) = (1-\alpha)\left(\frac{1}{\alpha}\frac{\tilde{w}(\varepsilon)}{\tilde{p}(\varepsilon)}\right)^{\frac{\alpha}{\alpha-1}}.$$

Her consumption in each state is constrained by the single budget constraint,

$$p(\varepsilon) c_3(\varepsilon) \leq p(\varepsilon)\Pi(\varepsilon) - \left(\frac{B - rM}{3}\right) + A_3(\varepsilon), \tag{40}$$

where

$$\left(\frac{B - rM}{3}\right) \equiv \mathcal{T}_3,$$

is her nominal tax liability.

Substituting Inequality (40) into (39) and using the no arbitrage condition and the definition of state prices leads to the entrepreneur's lifecycle constraint,

$$\mathbb{E}\left[\tilde{p}(\varepsilon) c_3(\varepsilon)\right] \leq \mathbb{E}\left[\tilde{p}(\varepsilon)\Pi(\varepsilon)\right] - \left(\frac{B - rM}{3}\right).$$

When the entrepreneur allocates consumption across states to maximize expected utility she will choose the following consumption demands,

$$c_3(\varepsilon) = \frac{\mathbb{E}\left[\tilde{p}(\varepsilon)\Pi(\varepsilon)\right]}{\tilde{p}(\varepsilon)} - \frac{1}{\tilde{p}(\varepsilon)}\left(\frac{B - rM}{3}\right). \tag{41}$$

Notice, and this is important, that dollar prices $p(\varepsilon)$ or $w(\varepsilon)$ no longer separately appear in the entrepreneur's budget constraint, which can be expressed entirely using state prices. Although the entrepreneur makes the same labor demand and output supply decisions in each state, she does not make the same consumption decisions. Instead of consuming the after tax profit in each state, access to an insurance market allows the entrepreneur to smooth consumption.

6.2. **Rational Expectations Equilibrium with Complete Participation.** In this sub-section we characterize the equations that define equilibrium in the complete participation economy. Recall that $\mathcal{W}$, the wealth of a worker, is defined as

$$\mathcal{W} = \mathbb{E}[\tilde{w}(\varepsilon)] + \left(\frac{B}{6} + \frac{rM}{3}\right).$$

Using this definition, the labor market equilibrium condition is given by Equation (42),

$$\left(\frac{1}{\alpha}\frac{\tilde{w}(\varepsilon)}{\tilde{p}(\varepsilon)}\right)^{\frac{1}{\alpha-1}} = 2 - \mu\frac{\mathcal{W}}{\tilde{w}(\varepsilon)}, \tag{42}$$

and the goods market equilibrium condition is,

$$\left(\frac{1}{\alpha}\frac{\tilde{w}(\varepsilon)}{\tilde{p}(\varepsilon)}\right)^{\frac{\alpha}{\alpha-1}} = \frac{1}{\tilde{p}(\varepsilon)}\left\{\mathbb{E}\left[\tilde{p}(\varepsilon)\Pi(\varepsilon)\right] - \left(\frac{B - rM}{3}\right)\right\} + \lambda\frac{\mathcal{W}}{\tilde{p}(\varepsilon)}. \tag{43}$$

Finally, equality of the demand and supply of money is represented by Equation (44),

$$\gamma\mathcal{W} = rM. \tag{44}$$

The important difference of the equations that characterize the complete participation economy from the incomplete participation economy, is to be found in Equation (43), which no longer contains terms in $w(L)$ or $w(H)$. That fact implies that equations (42) and (43) are identical in states L and H.

Given our amended definitions of demand and supply equations we are ready to define the concept of a complete participation rational expectations equilibrium.

Definition 3. *A complete participation rational expectations equilibrium is a monetary and a fiscal policy $\{M, B\}$, a belief function $\varphi(M, \varepsilon)$, an allocation $\{\{n_i(\varepsilon), M_i\}_{i\in 1,2}, \{c_i(\varepsilon)\}_{i=1,2,3}, y(\varepsilon), n(\varepsilon)\}_{\varepsilon\in\{H,L\}}$ and a set of state-dependent prices $\{\tilde{p}(\varepsilon), \tilde{w}(\varepsilon), Q(\varepsilon), r\}_{\varepsilon\in\{H,L\}}$ that satisfies budget balance and optimality. An equilibrium price system, $\{\tilde{p}(\varepsilon), \tilde{w}(\varepsilon), Q(\varepsilon), r\}_{\varepsilon\in\{H,L\}}$, is a non-negative 7-tuple such that equations (42) and (43) hold in each state, Equation (44) holds, and the money wage in each state is given by the belief function, Equation (29).*

Next, we turn to a proposition that characterizes the properties of employment, output and the distribution of output in the complete participation rational expectations equilibrium.

Proposition 4. *Let* $\{M,B\}\geq 0$ *characterize public sector policy, and let* $\{w(L), w(H)\}>0$ *be wages implied by a belief function* $\varphi(M,\varepsilon)$ *such that the feasibility hold*

$$w(\varepsilon) \geq \frac{\mu_i B}{4(1+\lambda+\mu)-6\mu_i}, \quad i\in\{1,2\}, \quad \varepsilon\in\{L,H\} \tag{45}$$

$$w(\varepsilon) \geq \frac{2-\mu+\lambda(2-3\alpha)}{\mu+\alpha\lambda}\frac{B}{2}, \quad \varepsilon\in\{L,H\}. \tag{46}$$

Define $q\equiv Q(L)/Q(H)$*. Then,*

$$q=\frac{w(H)}{w(L)}\frac{\pi_L}{\pi_H}. \tag{47}$$

The equilibrium value of $Q(H)$ *is given by the expression*

$$Q_H=\frac{M(6-2\gamma)\pi_H}{(1+q)(M(6-2\gamma)+B\gamma)\pi_H+6\gamma w(H)} \tag{48}$$

with

$$Q(L)=qQ(H) \qquad \text{and} \qquad Q=Q(L)+Q(H).$$

The equilibrium state wages $\tilde{w}(\varepsilon)$ *are given by the expressions*

$$\tilde{w}(L)=\frac{w(L)}{(1+q^{-1})\pi_L}, \qquad \tilde{w}(H)=\frac{w(H)}{(1+q)\pi_H}, \tag{49}$$

and the equilibrium state price of consumption goods $\tilde{p}(\varepsilon)$ *are equal to*

$$\tilde{p}(\varepsilon)=\frac{\tilde{w}(\varepsilon)}{\alpha}\left(2-\frac{\mu\mathcal{W}}{\tilde{w}(\varepsilon)}\right)^{1-\alpha}. \tag{50}$$

Conditional on the prices, $\tilde{w}(\varepsilon),\tilde{p}(\varepsilon)$ *and* $Q(\varepsilon)$ *characterized above, the equilibrium quantities* $c_i(\varepsilon), n_i(\varepsilon)$ *and* M_i *can be found from Equations (26) – (28), while* $n(\varepsilon), y(\varepsilon)$ *and* $c_3(\varepsilon)$ *are characterized in Equations (23) – (24) and (41).* □

See Appendix D for a proof of this proposition. We also have the following corollary,

Corollary 5. *Under full participation, the equilibrium associated with* any *belief function has the property that*

$$n_i(L)=n_i(H), \quad i\in\{1,2\} \qquad \text{and} \qquad c_i(L)=c_i(H), \quad i\in\{1,2,3\}.$$

Proof. The proof follows directly from the proposition above: the formula for $Q(\varepsilon)$ implies that state-wages $\tilde{w}(\varepsilon)$ are the same in both states, and thus, by definition, so are state-prices $\tilde{p}(\varepsilon)$. The solutions to the workers' optimization problems, equations (26) – (28), then show that the corresponding real allocations are state-invariant. This establishes that, in a complete participation rational expectations equilibrium, there is complete consumption and leisure insurance. ∎

To clarify what is happening in the model, we now characterize the entrepreneur's asset portfolio.

Proposition 6. *In the full participation model, the entrepreneur's asset position is given by*

$$A_3(H) = (1-\pi_H)\left(\frac{B-rM}{3}\right)\left(1-\frac{w(H)}{w(L)}\right), \quad A_3(L) = \pi_H\left(\frac{B-rM}{3}\right)\left(1-\frac{w(L)}{w(H)}\right).$$

Proposition 6 is proved in Appendix E.

An immediate implication of this proposition and the fact that $w(H) > w(L)$ is that $A_3(H)$ is negative, while $A_3(L)$ is positive. The entrepreneur uses the asset market to buy insurance from the workers against the $w(L)$ outcome and to sell insurance to the workers against the $w(H)$ outcome.

When the entrepreneur is excluded from trade in Arrow securities, she is positively affected by higher $w(\varepsilon)$ for two reasons. First, the entrepreneur pays for part of the nominal transfer which workers receive from the government. Higher nominal wages mean that the *real* value of the transfer is lower which makes her better off when $\varepsilon = H$. Second, the fact that workers are poorer in state $\varepsilon = H$ means that they consume less leisure. Equilibrium employment, output and the real value of profits are higher. In contrast, workers are worse off in state $\varepsilon = H$. Both groups of agents will trade Arrow securities up to the point at which their *real* consumption and leisure are constant across states.

6.3. **Nominal Bond and Equity Portfolios.** In this sub-section we translate the abstract notion of Arrow securities into the more familiar case in which agents cross-insure using debt and equity. We assume that a nominal bond pays a dollar in every state ε, while equities entitle their owners to a share of the entrepreneur's nominal profit stream, which we denote with the symbol $\tilde{\Pi}(\varepsilon)$ to distinguish it from the real profit stream, $\Pi(\varepsilon)$,

$$\tilde{\Pi}(\varepsilon) \equiv p(\varepsilon)\, y(\varepsilon) - w(\varepsilon) n(\varepsilon).$$

Using these definitions we prove the following proposition.

Proposition 7. *In the full participation model, the entrepreneur purchases nominal bonds with a face value of*

$$B_3 \equiv \frac{B-rM}{3},$$

where

$$r = \frac{1-Q}{Q}.$$

The purchase of bonds by entrepreneurs is financed by selling a share λ of the entrepreneur's profit stream where

$$\lambda = \frac{QB}{Q(L)\,\tilde{\Pi}(L) + Q(H)\,\tilde{\Pi}(H)}.$$

Proof. See Appendix F. ∎

If workers and firms were to trade two assets, debt and equity, the entrepreneur would use nominal bonds to insure herself against volatility in real tax expenditures. In equilibrium, fluctuations in the nominal price level cause fluctuations in the real value of tax liabilities that are perfectly insured by the nominal bonds she purchases from workers. Workers provide this insurance by purchasing a share in the firm. This share is risk free because fluctuations in the nominal profit stream are offset, in equilibrium, by fluctuations in the price level.

The equilibrium with complete participation Pareto dominates the equilibrium in the absence of complete participation because it provides an additional opportunity for trade. In Section 7 we show how the government can restore Pareto efficiency, even if entrepreneurs are not present in period 1, by trading on their behalf.

7. The Role of Qualitative Easing in a World of Incomplete Participation

We are now ready to discuss the role of qualitative easing; a policy in which the Treasury, and or the Central Bank, makes trades of debt for equity in the asset markets. In a complete markets environment, with complete participation, a policy of this kind will have no real effects (Woodford, 2012). We show that, in an environment with incomplete participation, central bank trades in the asset markets will improve welfare.

We return to the case where entrepreneurs are excluded from participating in asset trades, and we assume that the treasury makes dollar denominated lump-sum transfers worth $QB/2$ to workers, paid for by issuing nominal debt with a face value of B. We retain the assumption that workers trade two Arrow securities and, in addition we add the possibility of trading debt and equity. In this environment, there are redundant assets since the returns to debt and equity can be replicated by trades in Arrow securities.

A bond is a claim to B dollars in state ε that can be replicated by the purchase of B Arrow securities of type L and B securities of type H. Equity, is a claim to $\lambda\tilde{\Pi}(\varepsilon)$ dollars in state ε that can be replicated by the purchase of a portfolio of $\lambda\tilde{\Pi}(L)^{-1}$ securities of type L and $\lambda\tilde{\Pi}(H)^{-1}$ securities of type H where λ is a share of the firm.

We assume that workers purchase Arrow securities and they do not buy or sell bonds or equities.[12] We continue to assume that the central bank, purchases debt A_{CB} where,

$$M = QA_{CB}.$$

In addition, we allow the bank to make supplementary trades in debt and equity, subject to the constraint that these supplementary security purchases are self-financing,

$$M = QA_{CB} + Q\tilde{A}_{CB} + P_E\lambda_{CB}.$$

[12]This assumption is made for convenience. Because bonds and equities are redundant securities, the allocation of worker's assets across the two Arrow securities plus debt and equity is indeterminate.

The self-financing condition implies that

$$Q\tilde{A}_{CB} + P_E \lambda_{CB} = 0.$$

Here, $\tilde{A}_{CB}$ are additional purchases of debt by the central bank that may be positive or negative, λ_{CB} is the number of shares to the nominal profit stream that is bought or sold by the central bank and

$$P_E = Q(L)\tilde{\Pi}(L) + Q(H)\tilde{\Pi}(H),$$

is the price of a claim to a share in the firm. We allow short sales so that λ_{CB} may be negative.

Let $\mathcal{S}$ denote seigniorage revenues associated with money issuance and define

$$\tilde{\mathcal{S}}(\varepsilon) = \mathcal{S} + \left[\tilde{\Pi}(\varepsilon) - \tilde{A}_{CB}\right],$$

where

$$\left[\tilde{\Pi}(\varepsilon) - \tilde{A}_{CB}\right],$$

is the additional profit or loss associated with the risky component of the bank's balance sheet. Using these definitions, we arrive at the balance sheet of the central bank presented in Table 2.

Assets	Liabilities
QA_{CB}	M
$Q\mathcal{S}$	
$Q\tilde{A}_{CB}$	$P_E\lambda_{CB}$
	E_{CB}

TABLE 2. The Central Bank Balance Sheet

As in our previous model, the seigniorage revenues from money creation are repaid to the treasury. However, there is now risk associated with the central bank's balance sheet. The following modified definition of an equilibrium accounts for the fact that the central bank trades in the asset markets.

Definition 4. *A rational expectations equilibrium with a self-financing stabilization policy is a monetary and a fiscal policy $\{M, B, A_{CB}, \lambda_{CB}\}$, a belief function $\varphi(M, \varepsilon)$, an allocation $\{\{n_i(\varepsilon), M_i\}_{i\in 1,2}, \{c_i(\varepsilon)\}_{i=1,2,3}\ ,\ y(\varepsilon), n(\varepsilon)\}_{\varepsilon\in\{H,L\}}$ and a set of state-dependent prices $\{\tilde{p}(\varepsilon), \tilde{w}(\varepsilon), Q(\varepsilon), r\}_{\varepsilon\in\{H,L\}}$ that satisfies budget balance and optimality and the self-financing condition,*

$$Q\tilde{A}_{CB} + \lambda_{CB}P_E = 0,$$

where

$$P_E = Q(L)\tilde{\Pi}(L) + Q(H)\tilde{\Pi}(H).$$

An equilibrium price system, $\{\tilde{p}(\varepsilon), \tilde{w}(\varepsilon), Q(\varepsilon), r\}_{\varepsilon\in\{H,L\}}$, is a non-negative 7-tuple such that equations (42) and (43) hold in each state, Equation (44) holds, and the money wage in each state is given by the belief function, Equation (29).

Proposition 8, establishes that there exists a set of central bank trades that leads to the same real allocations as those in the complete participation equilibrium described in Proposition 4.

Proposition 8. *Let $\left\{M, B, \tilde{A}_{CB}, \lambda_{CB}\right\} \geq 0$ characterize public sector policy, and let $\{w(L),$
$w(H)\} > 0$ be wages implied by a belief function $\varphi(M, \varepsilon)$ such that the feasibility constraints hold*

$$\begin{aligned} w(\varepsilon) &\geq \frac{\mu_i B}{4(1+\lambda+\mu) - 6\mu_i}, \quad i \in \{1,2\}, \quad \varepsilon \in \{L,H\} \\ w(\varepsilon) &\geq \frac{2-\mu+\lambda(2-3\alpha)}{\mu+\alpha\lambda}\frac{B}{2}. \ \ \varepsilon \in \{L,H\} \end{aligned}$$

Let the central bank buy debt equal to $\tilde{A}_{CB}$, financed by selling equities, λ_{CB}, where,

$$\tilde{A}_{CB} = B - rM, \quad \textit{and} \quad \lambda_{CB} = -Q\left(\frac{B - rB}{Q(L)\tilde{\Pi}(L) + Q(H)\tilde{\Pi}(H)}\right). \tag{51}$$

The prices $Q(L)$, $Q(H)$ and $r = (1-Q)/Q$ and the money value of profits in each state $\tilde{\Pi}(L)$ and $\tilde{\Pi}(H)$, are the values defined in Proposition 6.

Under this policy, there exists a unique equilibrium in which allocations are the same as those implemented in the complete participation rational expectations equilibrium.

Proof. See Appendix G. ∎

The fact that the equilibrium allocations are identical to those under complete participation means that the central bank is able to restore efficiency. In the proof of the proposition we establish that the central bank's position in the asset markets is three times the position that would be taken by the entrepreneur in the counter-factual complete markets equilibrium.[13] Hence the workers' portfolios of risky assets will be larger under complete participation than without. In both cases, the real value of the workers' and entrepreneurs' after tax incomes will be stabilized under the optimal policy.

Corollary 9. *In the stabilization equilibrium, the return on a real indexed bond is equal to the real return from holding equity.*

Proof. In the Pareto efficient equilibrium the real value of profit is the same in both states. It follows immediately that the return to an indexed bond is the same as the return to an equity share . ∎

[13]This number follows from the fact that there are two times as many workers as entrepreneurs. In general, the size of the central bank portfolio will be a function of the population growth rate.

This corollary implies that, *when all uncertainty is non-fundamental*, the central bank can implement the optimal policy by standing ready to trade indexed bonds at the same price as claims to the stock market. By doing so, it would end up holding the optimal asset portfolio $\left\{\tilde{A}_{CB}, \lambda_{CB}\right\}$ described in Proposition 8.

8. Summary

Willem Buiter made the distinction between Quantitative Easing, defined as an increase in the size of the central bank balance sheet, and Qualitative Easing, defined as a change in its risk composition. In this paper we have provided a theory that explains the channel by which Qualitative Easing influences asset prices. According to our explanation, some asset price fluctuations are Pareto inefficient because people cannot insure themselves against the state of the world they are born into. By trading debt for equity in the asset markets, the central bank can make trades that stabilize Pareto inefficient asset price movements and that make everyone better off.

Our explanation builds on the idea that general equilibrium models of money always contain multiple equilibria. Standard accounts of asset market dynamics struggle to account for the volume of trades that we observe in real world asset markets (Milgrom and Stokey, 1982), and for the the observed volatility in asset prices, relative to dividend movements (Shiller, 1981). We explain these features of data by exploiting shifts across different equilibria in the presence of incomplete participation. In our view, significant portions of asset price fluctuations in the real world are caused by self-fulfilling shifts from one equilibrium to another that are associated with inefficient fluctuations in wealth.

Although we have explained our case in a simple two-period model, multiple equilibria are endemic to monetary models and our argument is more general than the model that we have used to explain it. The fact that asset price volatility is Pareto inefficient provides, we believe, a strong case to make Qualitative easing a permanent component of future financial policy.

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Appendix A. Proof of Proposition 1

Proof. Combining the definition of wealth (2) with the money market clearing condition, Equation (14), we have the following definition of wealth that must hold in equilibrium

$$\mathcal{W} = \frac{6w + B}{2\left(1 + \lambda + \mu\right)}. \tag{A1}$$

Feasibility requires that

$$n_i = 2 - \mu_i \frac{\mathcal{W}}{w} > 0,$$

for both types, which implies the first feasibility condition, Equation (A2).

$$w > \frac{\mu_i B}{4\left(1 + \lambda + \mu\right) - 6\mu_i}, \qquad i \in \{1, 2\} \tag{A2}$$

To derive the expression for the equilibrium value of r, we use the money market equilibrium condition, Equation (14), together with the equilibrium value of wealth from (A1). The real wage follows directly from inverting the labor market clearing equation.

For a valid equilibrium we also require that

$$c_3 = (1 - \alpha)\left(\frac{1}{\alpha}\frac{w}{p}\right)^{\frac{\alpha}{\alpha-1}} - \frac{(B - rM)}{3p} > 0.$$

Using equilibrium prices and wealth from equations (15), and the market clearing equations, (9) – (10), evaluated at equilibrium prices, we arrive at the second feasibility condition,

$$w \geq \frac{2 - \mu + \lambda\left(2 - 3\alpha\right)}{\mu + \alpha\lambda}\frac{B}{2}.$$

∎

Appendix B. Proof of Proposition 2

Proof. We begin with three facts that follow from the definitions of market clearing, Equations (26) – (28). First, from money market clearing,

$$\mathcal{W} = \frac{\left(1 - Q\right)M}{Q}\frac{1}{\gamma} \equiv \frac{x}{\gamma}, \tag{B1}$$

where we define

$$x \equiv \frac{(1-Q)\,M}{Q} \equiv rM. \tag{B2}$$

Second, putting together the labor market and goods market clearing equations for each state, equations (26) and (27), and using the definition of x from Equation (B2) and the money market clearing equation, (B1), we have that,

$$2 - \frac{\theta x}{\tilde{w}(\varepsilon)} = \frac{x}{3w(\varepsilon)} - \frac{B}{3w(\varepsilon)}, \;\; \varepsilon \in \{L, H\}, \tag{B3}$$

where we define

$$\theta \equiv \frac{\lambda + \mu}{\gamma}.$$

Third, we use the definition of $\tilde{w}(\varepsilon)$,

$$\tilde{w}(\varepsilon) \equiv \frac{Q(\varepsilon)\,w(\varepsilon)}{Q\pi_\varepsilon}, \;\; \varepsilon \in \{L, H\}. \tag{B4}$$

Substituting (B4) into (B3) gives,

$$2 - \frac{Q\pi_\varepsilon \theta x}{Q(\varepsilon)\,w(\varepsilon)} = \frac{x}{3w(\varepsilon)} - \frac{B}{3w(\varepsilon)}, \;\; \varepsilon \in \{L, H\}.$$

Rearranging this equation and using the no arbitrage condition, $Q = Q(L) + Q(H)$ leads to the following expression for x

$$x = \frac{[6w(L) + B]\,Q(L)}{Q(L) + 3\pi_L\theta\,[Q(L) + Q(H)]} = \frac{[6w(H) + B]\,Q(H)}{Q(H) + 3\pi_H\theta\,[Q(L) + Q(H)]}. \tag{B5}$$

Next define $q \equiv Q(L)/Q(H)$, and divide the numerator and denominator of Equation (B5) by $Q(H)$ to give

$$\frac{[6w(L) + B]\,q}{q + 3\pi_L\theta\,[1+q]} = \frac{[6w(H) + B]}{1 + 3\pi_H\theta\,[1+q]}. \tag{B6}$$

Rearranging this equation and defining

$$X_L \equiv [6w(L) + B], \;\; X_H \equiv [6w(H) + B], \;\; Y_L = 3\pi_L\theta, \;\; Y_H \equiv 3\pi_H\theta,$$

$$X_L q\,[1 + Y_H(1+q)] = X_H\,[q + Y_L(1+q)],$$

and

$$\theta_1 \equiv \frac{[X_H(1+Y_L) - X_L(1+Y_H)]}{X_L Y_H}, \;\; \theta_2 \equiv \frac{X_H Y_L}{X_L Y_H},$$

gives the following quadratic equation in q,

$$q^2 - \theta_1 q - \theta_2 = 0.$$

This is Equation (32) in Proposition 2. Let r_1 and r_2 be the roots of this quadratic and note that r_1 and r_2 are given by the expressions

$$r_i = \frac{1}{2}\left(\theta_1 \pm \sqrt{\theta_1^2 + 4\theta_2}\right).$$

It follows that both roots are real and that there is a unique non-negative root.

Next note that

$$x \equiv M\frac{(1-Q)}{Q} \equiv \left(\frac{1}{Q_H} - 1 - q\right)\frac{M}{1+q}, \tag{B7}$$

and use Equation (B6) to write

$$x = \frac{X_L q}{q + Y_L (1+q)}. \tag{B8}$$

Combining equations (B8) and (B7) leads to the expression for $Q(H)$, Equation (33) in Proposition 2. Equation (34) follows from the definition of Q. To derive Equations (35) use the definitions of $w(\varepsilon)$, Equations (17).

Equation (36) follows from the labor market clearing equation and (37) from the (B1) and the no arbitrage assumption. It remains to check that Inequality (30) is sufficient to guarantee that labor supply is feasible and that (31) guarantees that entrepreneurs' consumption is non-negative. That follows from the fact that theses assumptions guarantee feasibility in each state individually and therefore feasibility in a convex combination of the states as well. ∎

Appendix C. Proof of Corollary 3

Proof. Labor supply of person i, for $i \in \{1,2\}$ is given by the expression

$$n_i(\varepsilon) = 1 - \mu_i \frac{\mathcal{W}}{\tilde{w}(\varepsilon)},$$

and consumption by

$$c_i(\varepsilon) = \lambda_i \frac{\mathcal{W}}{\tilde{w}(\varepsilon)}.$$

Hence to establish inequalities (38) we need only show that

$$\tilde{w}(L) \neq \tilde{w}(H).$$

But from Equation (B3) we have that

$$\frac{\theta x}{\tilde{w}(\varepsilon)} = \frac{B}{3w(\varepsilon)} - \frac{x}{3w(\varepsilon)} - 2,$$

from which it follows that $\tilde{w}(L) = \tilde{w}(H)$ if and only if, $w(L) \neq w(H)$. The inequality of the consumption of entrepreneurs across states follows from the fact that their income is a function of the real wage. ∎

Appendix D. Proof of Proposition 4

Proof. Combining labor market equilibrium, Equation (42) with goods market equilibrium (43) leads to

$$\frac{1}{\alpha}\left(2 - \mu\frac{\mathcal{W}}{\tilde{w}(\varepsilon)}\right) = \frac{1}{\tilde{w}(\varepsilon)}\left\{\mathbb{E}\left[\tilde{p}(\varepsilon)\Pi(\varepsilon)\right] - \left(\frac{B-rM}{3}\right)\right\} + \lambda\frac{\mathcal{W}}{\tilde{w}(\varepsilon)}, \qquad \varepsilon \in \{L,H\}.$$

Because these two state equations are identical (the term in the wiggly brackets is a constant) it immediately follows that

$$\tilde{\omega}(L) = \tilde{w}(H) \equiv \tilde{w}. \tag{D1}$$

Using this fact, and the definition of $\tilde{w}(\varepsilon)$ gives,

$$q \equiv \frac{Q(L)}{Q(H)} = \frac{w(H)\pi_L}{w(L)\pi_H}.$$

This establishes the expression for q, Equation (47), in the statement of Proposition 4.

Next we seek expressions for $Q(L)$ and $Q(H)$ individually. Combining the definition of workers wealth, Equation (19) with the money market equilibrium condition, Equation (44), and using Equation (D1) gives the following equation linking $\tilde{\omega}$ and r,

$$\mathcal{W} = \frac{rM}{\gamma} = \tilde{w} + \frac{B}{6} + \frac{rM}{3} \tag{D2}$$

Note also that

$$\tilde{w} = \frac{w(H)Q(H)}{Q\pi_H}.$$

The no arbitrage condition, $Q = Q(L) + Q(H)$ implies that

$$\frac{Q(H)}{Q} = \frac{1}{1+q}.$$

Using no arbitrage and the definition of r we also have that

$$r = \frac{1-Q}{Q} = \frac{1-Q_L-Q_H}{Q_L+Q_H} = \frac{\frac{1}{Q_H}-(1+q)}{1+q},$$

which simplifies to give,

$$Q_H = \frac{1}{(1+r)(1+q)}. \tag{D3}$$

Next, we rearrange Equation (D2)

$$rM\left(\frac{3-\gamma}{\gamma}\right) = 3\frac{w(H)}{(1+q)\pi_H} + \frac{B}{2},$$

to find the following expression for $(1+r)$

$$(1+r) = \left(M\left(\frac{3-\gamma}{\gamma}\right) + 3\frac{w(H)}{(1+q)\pi_H} + \frac{B}{2}\right)\left(M\left(\frac{3-\gamma}{\gamma}\right)\right)^{-1}. \tag{D4}$$

Finally, combining (D4) with (D3) and using the definition of γ gives Equation (48) in the statement of Proposition 4, which is the expression we seek

$$Q_H = \frac{M(6-2\gamma)\pi_H}{(1+q)(M(6-2\gamma)+B\gamma)\pi_H + 6\gamma w(H)}.$$

Equations (49) follow immediately from the definitions of state wages and Equation (50) follows from labor market clearing. The feasibility conditions, Inequalities (45) and (46) guarantee that labor supply for each worker and the consumption of entrepreneurs are each non-negative in equilibrium. ∎

Appendix E. Proof of Proposition 6

Proof. From the entrepreneur's budget constraint, Equation (40),

$$p(\varepsilon)\,c_3(\varepsilon) \leq p(\varepsilon)\,\Pi(\varepsilon) - \left(\frac{B - rM}{3}\right) + A_3(\varepsilon)\,. \tag{E1}$$

From the solution to the entrepreneur's problem, we have that

$$c_3(\varepsilon) = \frac{\mathbb{E}\left[\tilde{p}(\varepsilon)\,\Pi(\varepsilon)\right]}{\tilde{p}(\varepsilon)} - \frac{1}{\tilde{p}(\varepsilon)}\left(\frac{B - rM}{3}\right).$$

But from Proposition 4, $\tilde{p}(\varepsilon)$ is the same in both states and thus,

$$c_3(\varepsilon) = \Pi(\varepsilon) - \frac{1}{\tilde{p}(\varepsilon)}\left(\frac{B - rM}{3}\right). \tag{E2}$$

Rearranging Equation (E1) and combining it with (E2) gives the following expression,

$$A_3(\varepsilon) = \left(\frac{B - rM}{3}\right)\left(1 - \frac{p(\varepsilon)}{\tilde{p}(\varepsilon)}\right). \tag{E3}$$

Finally, from the definitions of $\tilde{p}(\varepsilon)$ and q we have that

$$\frac{p(H)}{\tilde{p}(H)} = \frac{\pi_H}{1+q}, \qquad \frac{p(L)}{\tilde{p}(L)} = \frac{q\pi_L}{1+q}. \tag{E4}$$

Combining equations (E3) and (E4)and using the facts that

$$q = \frac{w(H)}{w(L)}\frac{\pi_L}{\pi_H} = \frac{w(H)}{w(L)}\frac{1-\pi_H}{\pi_h},$$

gives

$$A_3(H) = (1-\pi_H)\left(\frac{B - rM}{3}\right)\left(1 - \frac{w(H)}{w(L)}\right), \quad A_3(L) = \pi_H\left(\frac{B - rM}{3}\right)\left(1 - \frac{w(L)}{w(H)}\right),$$

which is the expression we seek. ∎

Appendix F. Proof of Proposition 7

Proof. By purchasing bonds with face value

$$\frac{B - rM}{3},$$

the entrepreneur consumes

$$c_3(\varepsilon) = \frac{\tilde{\Pi}(\varepsilon)}{p(\varepsilon)} - \frac{(B - rM)}{3p(\varepsilon)} + \left[\frac{B_3}{p(\varepsilon)} - \lambda\frac{\tilde{\Pi}(\varepsilon)}{p(\varepsilon)}\right],$$

where

$$A_3(\varepsilon) \equiv B_3 - \lambda\tilde{\Pi}(\varepsilon)\,,$$

is the dollar value of Arrow securities in state ε. This is equal to the face value of debt, B_3, minus the share of profits, $\lambda\Pi(\varepsilon)$ that was sold to finance the purchase of debt. We established in Proposition 4, that $\tilde{\Pi}(\varepsilon)/p(\varepsilon)$ is the same in both states. It follows that if

$$B_3 = \frac{B - rM}{3},$$

that the entrepreneurs consumption is independent of the state. The share of profits that the entrepreneur sells to workers, λ, is defined by the entrepreneur's budget constraint in period 1,

$$QB_3 - \lambda P_E = 0,$$

where

$$P_E = Q(L)\tilde{\Pi}(L) + Q(H)\tilde{\Pi}(H),$$

is the price of a claim to the money value of profits. ∎

Appendix G. Proof of Proposition 8

To prove this proposition we show first that, if the security prices $Q(L)$ and $Q(H)$ and the nominal profit streams $\tilde{\Pi}(L)$ and $\tilde{\Pi}(H)$ are equal to the equilibrium values defined in Proposition 6 then the portfolio defined by Equation (51) stabilizes the real value of tax revenues.

In state ε the tax revenue levied by the treasury is given by the expression,

$$\mathcal{T}(\varepsilon) = [B - \mathcal{S}] - \left[\tilde{A}_{CB} + \lambda_{CB}\tilde{\Pi}(\varepsilon)\right].$$

To stabilize the real value of tax revenues the central bank must take a net asset position such that

$$\frac{[B - \mathcal{S}] - \left[\tilde{A}_{CB} + \lambda_{CB}\tilde{\Pi}(L)\right]}{w(L)} = \frac{[B - \mathcal{S}] - \left[\tilde{A}_{CB} + \lambda_{CB}\tilde{\Pi}(H)\right]}{w(H)}. \tag{G1}$$

By holding additional bonds equal to

$$\tilde{A}_{CB} = [B - S],$$

Equation (G1) gives

$$\mathcal{T}(L) \equiv \lambda_{CB}\frac{\tilde{\Pi}(L)}{w(L)} = \lambda_{CB}\frac{\tilde{\Pi}(H)}{w(H)} \equiv \mathcal{T}(H).$$

But from by definition,

$$\tilde{\Pi}(\varepsilon) = \Pi(\varepsilon)\, p(\varepsilon) = \Pi(\varepsilon)\, w(\varepsilon)\,\frac{\tilde{p}(\varepsilon)}{\tilde{w}(\varepsilon)},$$

where the last equality follows since

$$\frac{p(\varepsilon)}{w(\varepsilon)} = \frac{\tilde{p}(\varepsilon)}{\tilde{w}(\varepsilon)}.$$

Combining these expressions gives

$$\mathcal{T}(L) \equiv \lambda_{CB}\Pi(L)\,\frac{\tilde{p}(L)}{\tilde{w}(L)} = \lambda_{CB}\Pi(H)\,\frac{\tilde{p}(H)}{\tilde{w}(H)} \equiv \mathcal{T}(H).$$

But from Proposition 6, $\Pi(H) = \Pi(L)$. Hence the portfolio

$$\tilde{A}_{CB} = B - rM, \quad \text{and} \qquad \lambda_{CB} = -Q\left(\frac{B - rB}{Q(L)\tilde{\Pi}(L) + Q(H)\tilde{\Pi}(H)}\right),$$

is self-financing and stabilizes the real value if tax revenues as claimed.

Next we establish that this tax policy generates the same after tax wealth positions for entrepreneurs and workers that they would choose if entrepreneurs could self insure. We showed in Proposition that entrepreneurs would choose to hold debt equal to

$$B_3 = \frac{B - rM}{3}.$$

In the counter-factual complete participation equilibrium the wealth of the entrepreneur is equal to

$$\Pi(\varepsilon) - \frac{1}{3}\left(\frac{B - rM}{w(\varepsilon)}\right) + \left(\frac{B_3 - \lambda P_E}{w(\varepsilon)}\right),$$

where

$$\frac{1}{3}\left(\frac{B - rM}{w(\varepsilon)}\right),$$

is the real value of her tax obligation and

$$\{B_3, \lambda P_E\},$$

is the debt and equity portfolio that she takes to offset fluctuations in after-tax wealth.

In contrast, in the equilibrium with policy stabilization, the after tax wealth of the entrepreneur is

$$\Pi(\varepsilon) - \frac{1}{3}\left(\frac{B - rM}{w(\varepsilon)}\right) + \frac{1}{3}\left(\frac{\tilde{A}_{CB} - \lambda_{CB} P_E}{w(\varepsilon)}\right).$$

It follows immediately that if the central bank chooses a policy where

$$\tilde{A}_{CB} = 3B_3 = B - rM,$$

then the after tax wealth of the entrepreneur is identical in the equilibrium with stabilization as in the counter-factual complete markets equilibrium. It follows from Walras law that stabilizing the entrepreneurs income at its complete participation value also stabilizes the workers wealth at its complete participation value.